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EDITED BY
FRANK MORLEY

WITH THE COÖPERATION OF
A. COHEN, CHARLOTTE A. SCOTT, A. B. COBLE
AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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Groups of Order $2^{m'}$ Which Contain a Relatively Large Number of Operators of Order Two.

By G. A. Miller.

§ 1. *Introduction*

When exactly one-half of the operators of a group are of order 2 the group is obtained by extending an abelian group of odd order by means of an operator of order 2 which transforms each operator of this abelian group into its inverse, and if a group is constructed in this manner then exactly one-half of its operators are of order 2. If more than one-half of the operators of a group are of order 2 this excess must always be a number of the form $2^a - 1$ and there is an infinite system of such groups for every positive integral value of a.*

From the fact that the number which expresses the excess of the operators of order 2 over half the order of the group is always of the form $2^a - 1$ it results directly that in a group of order $2^{m'}$ the number of operators whose orders exceed 2 is always equal to the order of the group multiplied by a number of the form $\frac{2^n - 1}{2^{n+1}}$ and there are groups in which this product gives the number of the operators whose orders exceed 2 for every positive integral value of n. The smallest relative number of operators whose order exceeds 2 in a group of order $2^{m'}$ is therefore one-fourth of the order of the group in case there is at least one such operator.

Let G be any group in which more than one-half of the operators are of order 2. When G is abelian it is evidently of type $(1, 1, 1, \cdots)$ and hence we shall assume in what follows that G is non-abelian. As more than one-half of the operators of G are of order 2 these operators generate G and hence some of the operators of order 2 contained G must be non-invariant. If we let H_1 represent the subgroup composed of all the operators of G which are commutative with one such non-invariant operator s_1 then more than one-half of the operators of H_1 must be of order 2.

If H_1 is non-abelian it contains a non-invariant operator s_2 of order 2, and we may represent by H_2 the subgroup of H_1 composed of all its operators which are commutative with s_2. The subgroup H_1 includes the central of G

* *Bulletin of the American Mathematical Society*, Vol. XXV (1919), p. 33).

and H_2 includes this central as well as s_1. If H_2 is non-abelian we can continue this process until we arrive at an abelian group H_m which includes the central of G and $s_1, s_2, \cdots, s_{m-1}$. The group H_m is evidently of type $1, 1, 1, \cdots)$. In the article to which we referred it was proved that $m = 1$ whenever the order of G is divisible by some odd prime number but it is possible to construct groups of order $2^{m'}$ in which m exceeds any given integer.

All the operators of G can be uniquely represented in the following form

$$H_m + H_m t_2 + H_m t_3 + \cdots + H_m t_r,$$

where it may be assumed without loss of generality that each of the operators $t_2, t_3, \cdots, t_r$ is of order 2. It is known that whenever the order of G is divisible by an odd prime number then exactly one-half of the operators in each of the co-sets $H_m t_a$, $2 \leqq a \leqq r$, are of order 2 and that in no case can more than one of these co-sets involve more operators whose orders exceed 2 than operators of order 2. In the following section we shall prove that when there is one such co-set and $r > 2$ then exactly one-fourth of its operators are of order two.

§ 2. *Less than one-half of the operators in one co-set are of order two*

The co-set in which less than one-half of the operators are of order 2 is characterized by the fact that it is composed of all the operators of G which are commutative with less than one-half of the operators of H_m. In particular, H_m contains an operator s_1 which transforms an operator t_1 of this co-set into its inverse and hence if H_1 is defined as above it does not include any operator of this co-set. Every operator of order 2 found in H_1 but not in H_m $r > 2$, is either commutative with t_1 or transforms t_1 into its inverse, for if the product of t_1 and this operator of order 2 has an order which exceeds 2 then this product is transformed into its inverse by s_1 and hence the said operator of order 2 and t_1 are commutative.

The operators of order 2 contained in H_1 but not in H_m must either generate H_1 or an abelian subgroup of type $(1, 1, 1, \cdots)$ and index 2 under H_1. The former of these alternatives is impossible since t_1 cannot be commutative with one-half of the operators of H_m. Otherwise one-half of the operators of the co-set containing t_1 would be of order 2. It therefore results that when $r > 2$ exactly one-fourth of the operators of H_m are commutative with t_1 and that the central of H_1 is a subgroup of index 2 under H_m. As exactly one-fourth of the operators of the co-set containing t_1 are of order 2 we have proved the following theorem. *Whenever one of the co-sets*

$H_m t_2, \cdots, H_m t_r$, $r > 2$, contains more operators of order exceeding 2 than of order 2 then exactly one-fourth of the operators of this co-set are of order 2.

It may be assumed without loss of generality that $H_m t_r$ is the co-set in which less than one-half of the operators are of order 2, and that H_1 is composed of the co-sets $H_m t_2, \cdots, H_m t_\delta$ in addition to H_m. When $\delta = 2$ the subgroup H_m is invariant under H_1 because it is of index 2. If $\delta > 2$ this subgroup is still invariant since it is generated by all the operators of order 2 found in H_1 and having more than two conjugates under H_1. It has therefore been proved that H_m is invariant under H_1 independently of the value of δ.

It is now easy to prove that H_m is also invariant under G. In fact, the operator s_1 could have been so selected that it would be commutative with all the operators of any one of the given co-sets except the co-set $H_m t_r$, and hence H_m is invariant under each of these co-sets. It therefore results that *whenever more than one-half of the operators in one of the co-sets $H_m t_a$, $2 \leqq a \leqq r$ have orders which exceed 2 then H_m is invariant under G.*

The quotient group G/H_m involves only operators of order 2 besides the identity and hence this quotient group must be abelian and of type $(1, 1, 1, \cdots)$. The operators of H_m which are commutative with t_r constitute the central of G and include the commutators of G which arise from operators in H_m. The former of the theorems involved in the last sentence results directly from the fact that these operators are also commutative with every operator in the co-sets $H_m t_a$, $2 \leqq a \leqq \delta$, while the latter is a consequence of the fact that each of the operators $t_2, \cdots, t_r$ transforms H_m into an automorphism of order 2. If the commutator of order 2 arising from t_a, $2 \leqq a \leqq r-1$, would not be commutative with t_r then $t_a t_r$ could not transform H_m according to an automorphism of order 2.

Since all the commutators arising from operators of H_m are found in the central of G and since each of the operators $t_2, \cdots t_{r-1}$ is commutative with exactly one-half the operators of H_m it results that all the commutators with respect to H_m arising from these operators must also arise from t_r. As only three such commutators of order 2 arise from t_r it results that $\delta = 2$ and $r = 4$. The central of G may have any order of the form $2^{m''}$, $m'' > 1$ and there is evidently one and only one such group for every possible value of m''. Hence the following theorem has been established. *There is one and only one group of order 2^a, $a > 5$, such that one of the co-sets with respect to H_m involves more operators whose orders exceed 2 than operators of order 2. This group contains an abelian subgroup of index 4 and of type $(1, 1, 1, \cdots)$ and a central of index 16. The central quotient group is abelian and of type $(1, 1, 1, \cdots)$, and the commutator subgroup is of order 4.*

When $a > 6$ this group is evidently the direct product of the group of order 64 which belongs to this system and the abelian group of order 2^{a-6} and of type $(1, 1, 1, \cdots)$. In the form in which G presented itself above $m = 2$ but it is possible to select for s_1 an operator of H_m so as to make $m = 1$. If this is done the order of H_m remains the same since one of the corresponding co-sets must have less than one-half of its of order 2. When half of the operators of each co-set must be of order 2 it is clear that the order of H_m is independent of the choice of the set of operators $s_1, s_2, \cdots s_m$. Hence it results from the property of the groups just considered that *the order of H_m is an invariant of every group in which at least one-half of the operators are of order 2 and $r > 2$.*

It should also be noted that when $r > 2$ and G has a co-set, with respect to H_m, in which less than one-half the operators are of order 2 it must have such a co-set with respect to every possible H_m. Hence the possession of such a co-set is an invariant property of the group. When $r = 2$ this is clearly not the case, but when this condition is satisfied m is an invariant, being equal to unity, and H_1 is one of two groups. One of these must be of index 2 while for the other r may exceed 2.

If both of these subgroups are of index 2 exactly one-fourth of the operators of G have an order greater than 2 and G is known to be the direct product of the octic group and an abelian group of order 2^a and of type $(1, 1, 1, \cdots)$. When one of them has an index greater than 2 then each of the co-sets which arise with respect to this subgroup has exactly half of its operators of order 2 while the remaining operators of this co-set are of order 4 and have a common square. The co-set with respect to the other of these two subgroups contains more operators of order 4 than of order 2, the number of the operators of order 2 being half the order of the smaller subgroup which may be used for H_1. The least order which this subgroup can have when m' is odd is $\frac{1}{2}(m' - 1)$. When m' is even it is $\frac{1}{2}(m' - 2)$ since the commutator subgroup appears in the central of G.

In this system of groups for which r may be equal to 2 it is evident that there is an infinite number of groups in which the number of operators is equal to the order of the group multiplied by a number of the form $\frac{2^n - 1}{2^{n+1}}$, where n is an arbitrary positive integer. In fact, if one such group is found we can find an infinitude of others by forming the direct product of this group and abelian groups of order 2^a and of type $(1, 1, 1, \cdots)$, where a represents a positive integer but is not otherwise limited.

§ 3. *Exactly one-half of the operators of each co-set are of order 2 and* $m > 1$

When $m > 1$ and exactly one-half of the operators of each co-set are of order 2 it is easy to see that it is not possible that the subgroup of index 2 under H_m composed of all its operators which are commutative with each of the operators of one co-set is the same for all the different co-sets. Let t be any operator of order greater than 2 contained in G and let s_1 be any operator of H_m which is commutative with t but not contained in the central of G. Let H_1, as before, represent the subgroup composed of all the operators of G which are commutative with s_1.

The product obtained by multiplying any operator of order 2 contained in H_1 into an arbitrary operator whose order exceeds 2 in $G - H_1$ is transformed into its inverse by s_1 whenever the order of this product exceeds 2. In this case the two factors of this product must be commutative. As every operator whose order exceeds 2 in $G - H_1$ is transformed either into itself or into its inverse by every operator of order 2 in H_1 it results that all the operators of H_1 transform every operator of $G - H_1$ whose order exceeds 2 either into itself or into its inverse. The squares of each of the operators of H_1 is therefore contained in the central of G. In particular, t^2 is in this central. As t was any operator of order greater than 2 contained in G it results that *the squares of all the operators of* G *are found in the central of* G *and hence* G *involves no operator whose order exceeds* 4.

Since the quotient group of G with respect to its central involves only operators of order 2 besides the identity this quotient group must be abelian. In particular, both H_1 and H_m must be invariant subgroups of G. Every non-invariant operator of order 2 contained in G has just two conjugates. For, if such an operator s_1 had more than two conjugates, it may be supposed to appear in H_m. It may be assumed that s_1 is transformed into itself multiplied by the three distinct operators s_0, s'_0, $s_0 s'_0$ of order 2 contained in the central of G. Let t_0, t'_0, $t_0 t'_0$ be three operators of G which effect these transformations respectively. If t_0 and t'_0 were not commutative with the operators of the same subgroup of index 2 under H_m there would be a co-set corresponding to H_m which would have less than half of its operators of order 2.

Hence it may be assumed that t_0 and t'_0 are commutative with the operators of the same subgroup of index 2 under G. There must be an operator t''_0 in G which is not commutative with all the operators of this subgroup. The operators of H_m which are commutative with both of the operators t_0 and t''_0 constitute a subgroup of index 4 under H_m and include the central of G. Hence t_0 and t''_0 are non-commutative with the operators of a common co-set

with respect to this subgroup and it may be assumed without loss of generality that they transform the operators of this co-set into themselves multiplied by two distinct operators since t_0 may be replaced by t'_0. Hence we are again led to a co-set with respect to H_m which involves more operators whose orders exceed 2 than operators of order 2. It has therefore been proved that when $m > 1$ for every possible choice of s_1, and one-half of the operators of each co-set with respect to H_m are of order 2 then each non-invariant operator of order 2 contained in G has exactly two conjugates under G.

From the fact that each non-invariant operator of order 2 contained in G has exactly two conjugates under G it is easy to determine additional fundamental properties of G. In particular, it is possible to select a set of independent generators of G in such a way that all of these generating operators are of order 2 and that each one is commutative with all of the others with the exception of at most one of them. In fact, we may select s_1 as the first independent generator of G and then let s_2 be any operator of order two in $G - H_1$. The operators common to the two subgroups composed of all the operators of G which are commutative with s_1 and s_2 respectively constitute a subgroup of index 4 under G which includes the central of G. The order of the product of s_1 and s_2 is 4.

If this subgroup of index 4 is abelian it must be of type $(1, 1, 1, \cdots)$ and H_1 must be abelian and of the same type. In this case m would be equal to 1 and hence we may assume that the subgroup in question is non-abelian. More than one-half of its operators must be of order 2 since more than one-half of the operators of G are of order 2. It is therefore possible to find two additional non-commutative generators of order 2, s_3 and s_4, such that each of these generators is commutative with both of the operators s_1, s_2 and that $(s_3 s_4)^2 = (s_1 s_2)^2$. If the last condition were not satisfied s_1, s_3 would be an operator of order 2 having more than two conjugates under G.

When the group composed of all the operators of G which are commutative with each of the four operators s_1, s_2, s_3, s_4 is abelian and of order greater than 2 it is of type $(1, 1, 1, \cdots)$ and G is the direct product of a subgroup of index 2 under this abelian group and the group of order 32 generated by the given four operators. On the other hand, when the operators of G which are commutative with each of the four operators s_1, s_2, s_3, s_4 are not all commutative with each other then the subgroup generated by these operators must involve two non-commutative operators of order 2, s_5 and s_6, such that the square of their product is equal to $(s_1 s_2)^2$.

This process can evidently be continued until we obtain either a set of generators of G or an invariant subgroup of G of order 2^{2a+1} such that G is

the direct product of this subgroup and an abelian group of type $(1, 1, 1, \cdots)$. Hence we have established the following theorem: *When m exceeds unity for every possible choice of H_m and one-half of the operators in each co-set of G with respect to H_m are of order 2 then G is generated by β operators of order 2 such that each of them is commutative with all of the others except at most one of them and the square of the product of every pair of non-commutative operators of this set is the commutator of order 2 contained in G.*

§ 4. *Exactly one-half of the operators of each co-set are of order 2 and $m = 1$*

Two cases require consideration according as $m = 1$ for every possible choice of operators from G for s_1 or as $m = 1$ for some one but not for every possible such choice. We shall first restrict our attention to the former of these cases and hence it results that the subgroup of index 2 under H_1 composed of all the operators of H_1 which are commutative with the operators of one of the $r - 1 > 1$ co-sets constitute the central of G.

Let t_1 and t_2 be two operators of order 2 found in any two distinct co-sets with respect to H_1. If t_1 and t_2 were commutative the central of G together with t_1 and t_2 would generate an abelian group of type $(1, 1, 1, \cdots)$ and of a larger order than the order of H_1. By using t_1 instead of s_1 we would therefore find a new group for H_1 which would give rise to a co-set in which less than one-half of the operators are of order 2. As this case has been considered it may be assumed that t_1 is not commutative with any operators of order 2 except those found in the central of G or in the co-set to which t_1 belongs.

Since t_1 transforms into its inverse each of the products obtained by multiplying t_1 successively into all the operators of order 2 contained in G, and as these products include all the operators of G whose orders exceed 2 it results that every operator G whose order exceeds 2 is transformed only into itself or into its inverse under G and that it must be transformed into itself by all the operators of G whose order exceeds 2 since it is transformed into its inverse by a number of operators of order 2 equal to one-half of the order of G. Hence it results that the operators whose orders exceed 2 in G generate an abelian group and that *G is either the dihedral or the extended dihedral group.*

It remains to consider the case when $r > 2$ and H_1 involves an operator s_1 which transforms every operator whose order exceeds 2 in G into its inverse and when, moreover, G contains a non-invariant operator s_2 which is commutative with operators whose orders exceed 2 in G. If the central of G is composed of half of the operators of H_1 then s_2 must be contained in an

abelian group of type $(1, 1, 1, \cdots)$ whose order exceeds that of H_1. As one of the co-sets of G with respect to this abelian group would contain more operators whose orders exceed 2 than operators of order 2 it may be assumed that the order of the central of G is less than one-half of the order of H_1. It therefore results that the square of every operator of G is found in the central of G and hence H_1 is an invariant subgroup of G.

We may form H_m, $m > 1$, by starting with s_2 instead of with s_1. Let H'_2 be the subgroup of G composed of all its operators which are commutative with s_2. The product of an operator t_1 of order 4 in H'_2 and an operator t_2 of order 4 in $G - H'_2$ is transformed into its inverse by each of the operators s_1 and s_2 whenever the order of this product exceeds 2. In this case, it results therefore from a transformation by s_1 that $t_1^{-1} t_2^{-1} = t_2^{-1} t_1^{-1}$ and hence t_1 and t_2 are commutative. On the other hand, it results from a transformation by s_2 that $t_1 t_2^{-1} = t_1^{-1} t_2^{-1}$. As the latter is impossible it follows that the product of an operator of order 4 in H'_2 and an operator of this order in $G - H'_2$ must be of order 2.

As t_2 is commutative with one-half of the operators of H_m and is transformed into its inverse by the other half it results that it is commutative with all the operators of order 2 in $H'_2 - H_m$. Since t_2 is any operator of order 4 in $G - H'_2$ all of these operators are commutative with the operators of the same subgroup of index 2 under H_m. This is contrary to the hypothesis that s_2 is commutative with t_1 but not with t_2. Hence it results that *if exactly half the operators in each co-set with respect to every possible H_m are of order 2 and $m = 1$ for one choice of s_1 it must be equal to unity for every such possible choice.* Hence the central of G is of index 2 under H_1.

§ 5. *Conclusion*

The most familiar system of groups in which at least one-half of the operators are of order 2 are the dihedral and the generalized dihedral groups and when the order of a group is divisible by an odd prime number there are no other groups which have the property that at least one-half of their operators are of order 2. When the order of G is $2^{m'}$ there are other infinite systems of groups such that the number of operators of order 2 in the group exceeds half the order of the group but in all these groups the order of each operator is a divisor of 4. The dihedral and the generalized dihedral groups may be characterized by the facts that at least one-half of their operators are of order 2, that all the operators of such a group which are commutative with an arbitrary non-invariant operator of order 2 constitute an abelian group of

type $(1, 1, 1, \cdots)$ and that they do not contain an abelian group of this type which is of even order and of index 2 unless the central is of index 4 or of index 1.

One of the simplest systems of groups which are neither dihedral nor extended dihedral but have the property that more than one-half of their operators are of order 2 is obtained by extending the abelian group of order $2^{m'-1}$ and of type $(1, 1, 1, \cdots)$ by means of an operator of order 2 which transforms this abelian group into itself but is commutative with less than one-half of its operators. The order of the central of such a group cannot be less than $\frac{1}{2}(m'-1)$ since the multiplying group must appear in the central. Hence these groups are possible when and only when $m' > 4$ and the number of such groups for a given value of m' is $\frac{1}{2}(m'-1)-1$ when m' is odd and $\frac{1}{2}(m'-2)-1$ when m' is even. It may be noted that for each of these groups $m = 1$, but that r has two possible values.

As a second system composed of groups which are neither dihedral nor extended dihedral but have the property that more than one-half of their operators are of order 2 we may consider those for which one co-set with respect to every possible H_m has fewer operators of order 2 than of larger order. The smallest group of this system is of order 64 and every other group of the system is the direct product of this group of order 64 and an abelian group of order 2^a and of type $(1, 1, 1, \cdots)$. Exactly 7/16 of the operators of each of these groups are of order 4.

Finally, there is a third system of groups which are neither dihedral nor extended dihedral but contain more operators of order 2 than operators of larger order. This system is characterized by the facts that half of the operators in each co-set with respect to H_m are of order 2 and $m > 1$. The smallest order of such a group is 2^5 since if it is not dihedral or generalized dihedral it must contain at least two pairs of non-commutative generators when it is generated by a set of $\beta \leqq 4$ operators of order 2 such that each operator of this set is commutative with all except at most one other operator of the set. Every non-invariant operator of such a group has two conjugates under the group. By adding to the three systems just described the system composed of the dihedral and the extended dihedral groups we obtain the four possible systems of groups of order $2^{m'}$ which involve all the groups whose order is of this form and which have the property that more than one-half of their operators are of order 2.

To exhibit an important analogy between the second and the third system described above it may be desirable to note that the second system may be characterized by the fact that each of its groups may be generated by a set of

$\beta \geqq 4$ operators of order 2 such that each of these operators is commutative with all of the others except at most one of them and that there are exactly two pairs of non-commutative operators in the set, whose products are of order 4 and have different squares. This system can also be defined by the fact that each of its groups contains the direct product of two octic groups, and when its order exceeds 64 it is the direct product of this group of order 64 and an abelian group of type $(1, 1, 1, \cdots)$.

The Green's Function for a Plane Contour.

By Hobart Dickinson Frary.

A Green's function is a function of a boundary and two points. The points, of which one is considered fixed and the other variable, both lie inside (or both outside) the boundary. In two dimensions the boundary is a contour. A large part of mathematical physics is devoted to the solution of boundary problems in two dimensions. All such problems can be solved immediately if the Green's function for the given boundary or contour is known.

The character of the Green's function depends on three things,—(1) the differential equation of which it is a solution, (2) the contour or boundary to which it applies, and (3) the boundary conditions. The class of Green's functions to which we shall confine ourselves in this article is the one of most frequent occurrence in mathematical physics, namely, the Green's function for Laplace's differential equation, with what Hilbert calls boundary condition (I); that is, $G = 0$ on the boundary. Having found this particular solution G, we can solve the problem of Dirichet in the plane; that is, we can determine a solution u satisfying any given condition of the form $u = U(\theta)$ on the boundary. This we can do by making use of the formula

$$u = 1/2\pi \int_c U \frac{\partial G}{\partial n} ds,$$

which in polar co-ordinates becomes

$$u(r, \theta) = 1/2\pi \int_0^{2\pi} U(\theta) \left[-r \frac{\partial G}{\partial r} d\theta + \frac{\partial G}{r \partial \theta} dr \right]$$

There are two well recognized methods of obtaining the Green's function in two dimensions, both of which, however, are very limited in their applications. These methods are the following:

(1) *The method of images.* This method is easily applicable to the circle, semi-circle, infinite strip, half plane, equilateral triangle, 30 degree right-triangle, and the rectangle; but for other contours it would involve complicated Riemann surfaces and hyper-elliptic functions. Apparently no one has attempted to apply the method to other polygons.

(2) *The method of Schwarz.* Schwarz obtained a general formula $w = f(z)$, where $f(z)$ is in the form of a definite integral, by means of which

any given polygon can be mapped conformally on to the unit circle. Knowing this mapping function $f(z)$, we can obtain the Green's function for the polygon immediately from the formula:

$$G = -Re \log f(z) \qquad (Re \equiv \text{real part of})$$

By this method we are able to get the Green's function for the equilateral triangle, the 45 degree right-triangle, an infinite strip, a regular five-pointed star, and a rectangle. Beyond this it is difficult to go. Other polygons lead to Abelian integrals and hyper-elliptic functions. Moreover, except for regular polygons, it seems to be impossible to determine the unknown constants which appear under the integral sign in the function $f(z)$. For regular polygons, however, we could evaluate Schwarz' integral,

$$w = \int_0^z \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

in the form of an infinite series, and take the real part of the logarithm. In other words, the series obtained by expanding

$$G = -Re \log \int_0^z \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

gives us the value of the Green's function for a regular polygon of n sides in a circle of radius

$$r_0 = \int_0^1 \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

where the point of discontinuity is at the center of the polygon. This work of Schwarz has been considerably discussed and amplified by other writers, particularly in a recent book by Study.* However, very little of fundamental importance has been added to Schwarz' original memoirs.

The method of arriving at the Green's function which we shall employ in this article is new, although in principle similar to the methods used by Fourier and Neuman in attacking various physical problems. We select an infinite set of linearly independent functions which are solutions of the given differential equation, and seek to expand the Green's function, or rather, the quantity $G - \gamma$, where γ is the principal solution ($\log 1/r$ in this case), in a series of these functions, determining the coefficients of the series so that the boundary conditions are satisfied. The determination of these coefficients depends upon the solution of an infinite set of equations in an infinity of unknowns. In the problems solved by Fourier the solution of the infinite

* *Vorlesungen über Ausgewählte Gegenstände der Geometrie*, 2tes Heft.

set of equations was simple. In the problem before us the solution is much more involved, though entirely manageable in a large number of cases.

This method of procedure is much more widely applicable than either of the two methods described above. These two methods give workable results for, at most, the regular polygons, two other polygons, the five-pointed star, and the rectangle. Our method is applicable, not only to all regular polygons, but to a large class of irregular polygons as well, and to many other contours. It gives the Green's function in the form, log $1/r$ plus an expansion in circular harmonics, $r^n \cos n\theta$ and $r^n \sin n\theta$. The coefficients of this expansion are each given in the form of an infinite series of constants and can therefore be determined to any desired degree of accuracy by taking a sufficient number of terms. We are convinced that the method is actually workable in a practical sense in a large number of cases.

We have confined our attention to a single class of Green's functions, but the same procedure might be employed to get the Green's function for any differential equation with any given boundary conditions.* It might also be extended to problems in three dimensions.

Formulation of the Problem.

By a plane contour we mean any closed plane curve which is regular in the sense of Osgood; that is, is composed of a finite number of analytic arcs or straight segments. An important case is that of the polygon.

The Green's function for any such contour is given by the following formula in polar co-ordinates:

$$G(r,\theta) = -\log r + u(r,\theta), \tag{1}$$

where the function u is harmonic, that is, satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

and has no singularities in the region.

Consider this function, $u(r,\theta)$. We know that it is everywhere harmonic in the area S enclosed by the contour C and has no singularities in S. Moreover, on the boundary, for which $r = R(\theta)$ and $u = U(\theta)$, we have $G = 0$ and

$$U(R,\theta) = \log R(\theta). \tag{2}$$

Draw any circle, radius r_0 (Fig. 1), about 0 so that it lies wholly in S. By a

* See Hadamard's Columbia lectures, pp. 47-52.

well-known theorem we can expand the harmonic function u uniquely in the series,

$$u = A_0/2 + \sum_1^\infty r^n (A_n \cos n\theta + B_n \sin n\theta), \tag{3}$$

Where A_n and B_n are the Fourier constants for the area enclosed by the circle: $0 \leqq \theta \leqq 2\pi$, $0 \leqq r \leqq r_0$. The functions $r^n \cos n\theta$ and $r^n \sin n\theta$ form a complete set of orthogonal functions for the circular boundary. Let us write equation (3) for convenience in the form

$$u = \sum_{m=0}^\infty c_m \phi_m(r, \theta), \tag{4}$$

where $\phi_0 = 1$, $\phi_1 = r\cos\theta$, $\phi_2 = r\sin\theta$, ...

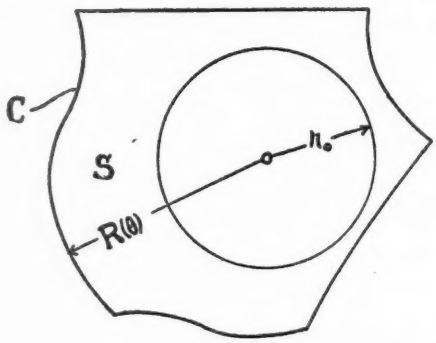

FIG. 1.

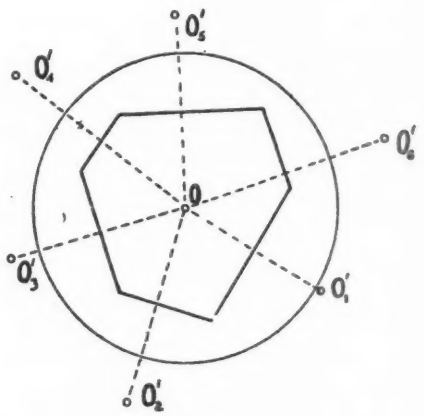

FIG. 2.

We can continue the function u analytically across the boundary C (Fig. 1) by means of Schwarz' principle of symmetry. The extended function thus obtained takes values equal but opposite in sign at every pair of points which are symmetrical * with respect to an arc or segment of the contour. At the symmetrical points or images $0'$ of the point 0, the function becomes infinite as $\log r$, but at all other symmetrical points it is harmonic. Hence if we construct the images † of the point 0 in the several arcs or segments of the contour, we know that the function u exists and is harmonic at least in a circle extending to the nearest image. Fig. 2 is drawn for a special case where the contour is a polygon.

Now, the question arises: Is the function u represented by the same

* For definition see Osgood, p. 671.

† For the method see Lery's thesis on *La Fonction de Green pour une Contour Algébrique*, p. 49.

expansion in the whole circle? To answer this question we cite the following theorem due to Bôcher.*

Theorem: If u is a function harmonic throughout the neighborhood of the point x_0y_0 and if, when this function is continued analytically, the distance from x_0y_0 to the nearest singular point of u, which lies in the same sheet of the Riemann surface generated by the analytic continuation in which x_0y_0 lies, is K, then the development† in polar co-ordinates (the point x_0y_0 being taken as pole),

$$u = A_0/2 + \Sigma\,(A_n r^n \cos n\theta + B_n r^n \sin n\theta),$$

converges and represents u throughout the interior of the circle of radius $r = K$ and does not converge throughout any continuum which does not lie in this circle.

If we restrict the problem, then, to contours of such a nature that a point 0 can be found whose images all lie outside of a circle about 0 enclosing the contour (this we can make the unit circle), we have from the above theorem, for all contours of this type and all such points 0 within each contour, the Green's function for the point 0 and the given contour in the form

$$\begin{aligned} G &= -\log r + A_0/2 + \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \\ &= -\log r + \sum_0^\infty c_m \phi_m(r,\theta). \end{aligned} \tag{5}$$

This expansion holds for all points inside the polygon and on the boundary. Moreover it is unique and we must be able to determine the constants A_n and B_n or c_m from the boundary conditions.

For the boundary we have $r = R(\theta)$ and $G = 0$, that is,

$$A_0/2 + \sum_1^\infty [A_n R^n(\theta) \cos n\theta + B_n R^n(\theta) \sin n\theta] = \log R(\theta)$$

or

$$\sum_0^\infty c_m \phi_m(\Phi) = \log R(\theta), \tag{6}$$

where $\Phi_0 = 1$; $\Phi_{2n-1} = R^n \cos n\theta$; $\Phi_{2n} = R^n \sin n\theta$.

Now we can apply a method of analytic continuation due to Borel, to obtain a new expression for the function u which will hold for a much more extended region, as follows: Equation (3) may be written

$$u = A_0/2 + Re \sum_1^\infty (A_n - iB_n)\, z^n,$$

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* Trans. Am. Math. Soc., Vol. 10, p. 278.

† Bôcher gives here also an equivalent in terms of polynomials in x and y.

where z is a complex variable. Since u is harmonic the expression $\sum_1^\infty (A_n - iB_n)\, z^n$ is analytic in a circle extending to the nearest image $0'$ of 0. Consider for example the polygon * $ABCDE$ (Fig. 3) with the point of discontinuity at a point 0 inside it. By means of Borel's integral we can continue the function $\sum_0^\infty A_n z^n$ analytically to the region enclosed by the polygon

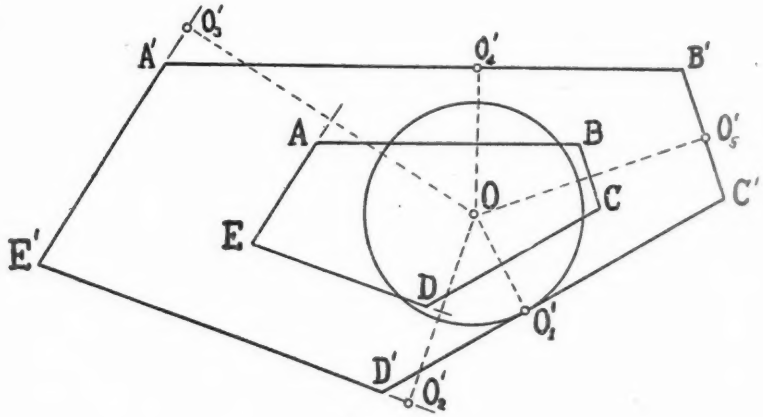

FIG. 3.

$A'B'C'D'E'$ whose sides are lines through the several images $0'$ and perpendicular respectively to the several lines $00'$. The expression for the function thus extended is

$$f(z) = \int_0^\infty e^{-t} \sum_0^\infty \frac{A_n (zt)^n}{n!}\, dt.$$

Applying this to the problem in hand, we have, in the real plane,

$$u = A_0/2 + Re \int_0^\infty e^{-t} \sum_1^\infty \frac{(A_n - iB_n)\ (zt)^n}{n!}\, dt$$

$$= A_0/2 + \int_0^\infty e^{-t} \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!}\, dt.$$

This function is harmonic inside the polygon $A'B'C'D'E'$ whose sides are respectively parallel to the sides of $ABCDE$. Hence this gives us a formula,

$$G = -\log r + A_0/2 + \int_0^\infty e^{-t} \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!}\, dt, \quad (7)$$

* Some or all of the sides of the polygon could be replaced by analytic arcs.

for the Green's function for any convex polygon whatever, with the point of discontinuity at any point whatever inside the polygon. Evidently this method can be applied, not only to polygons, but to *any convex contour whatever* and the point of discontinuity may be taken at *any point whatever* inside the contour.

The whole problem is then reduced to the method of determining the infinite set of constants A_n and B_n, or c_m. This problem may be solved in two ways: (1) by integral equations and (2) by an infinite set of algebraic equations. The first method gives only a formal solution; the second is a practical method as we shall show.

Formal Solution by Integral Equations.

The c's are Fourier constants. Hence, by a well-known theorem of Fischer and Riesz, the sum of their squares $\sum_0^\infty c^2_m$, converges. Moreover, if we select any set of orthogonal functions whatever, linear, planar or solid, for example $h_m(\chi)$, there exists one and only one function $H(\chi)$ for which c_m are the Fourier constants with respect to this orthogonal set. Whether the series thus obtained converges or actually represents the function at all is immaterial; for in any case the formulas for the coefficients hold, namely:

$$c_m = \int_{\chi_1}^{\chi_2} H(\chi)\ h_m(\chi)\ d\chi, \tag{8}$$

where the definite integral is either single, double or triple, according as the orthogonal functions chosen are linear, planar or solid. Now substitute these constants back in equation (6) and we have, provided the series in the bracket converges,

$$\int_{\lambda_1}^{\lambda_2} H(\chi) \left[\sum_0^\infty h_m(\chi)\, {}_m\Phi_m(\theta) \right] d\chi = \log R(\theta), \tag{9}$$

an integral equation of the first kind with unsymmetric kernel $K(\chi, \theta) = \sum_0^\infty h_m(\chi)\Phi_m(\theta)$. This integral equation has been studied by Picard, Schmidt, Bateman and others, and existence theorems have been worked out; but no general method of solution has yet appeared in the mathematical literature. The following special form of the equation, however, is of considerable interest:

If we take for the set of orthogonal functions $h_m(\chi)$ the linear set $\cos n\chi$,

$\sin n\chi$, the function $H(\chi)$ is nothing else than the value of u on the unit circle,

$$H(\chi) = u(1, \chi);$$

and the kernel reduces to the well-known expansion for

$$\frac{1 - R^2}{1 - 2R\cos(\chi - \theta) + R^2}. \qquad R < 1$$

Thus the integral equation (9) now becomes

$$1/2 \int_0^{2\pi} u(1, \chi) \frac{1 - R^2(\theta)}{1 - 2R(\theta)\cos(\chi - \theta) + R^2(\theta)} d\chi = \log R(\theta), \qquad (10)$$

where the left hand member is Poisson's integral. Hence we may state the solution of the problem before us as follows:

Consider a closed contour of such a nature that a circle, with center at a point 0, can be drawn completely enclosing it, but not enclosing any of the images of 0 with respect to the contour or arcs composing the contour. Make this the unit circle. Then the Green's function for this contour is obtained by using Poisson's integral backwards, as it were, to determine the value on the unit circle, namely $H(\chi) = u(1, \chi)$, of a harmonic function $u(r, \theta)$ which reduces to $\log R(\theta)$ for points of the given contour. Having found H we determine the Fourier constants or coefficients corresponding to it and substitute them in formula (5) to get the Green's function.

On account of the complicated character of the kernel in equation (10) the solution of this equation seems to be impracticable. We shall therefore take up in the following pages an entirely different method for determining the constants for equation (5.)

Practical Solution by an Infinite Set of Equations.

Going back to equation (6), it is possible to reduce the problem of determining the unknown constants c_m to the solution of an infinite number of equations for this infinite set of unknowns. One way to do this would be to expand the various powers of $R(\theta)$ as well as the function $\log R(\theta)$ in Fourier series and compare the coefficients of $\cos n\theta$ and $\sin n\theta$ on each side of the equation. This leads immediately to an infinite set of equations for the constants c_m, but these equations are very awkward to handle.

In order to get a more easily manageable set of equations we will construct a normalized, orthogonal set of functions from the set

$$\Phi_0 = 1,\ \Phi_1 = R(\theta)\cos\theta,\ \Phi_2 = R(\theta)\sin\theta,\ \Phi_3 = R^2(\theta)\cos 2\theta,\ \cdots\cdots,$$

functions which will be orthogonal for the variable θ between 0 and 2π. These functions we can construct by the method of Goursat, provided the functions are linearly independent. We shall first investigate this matter of linear independence.

An infinite set of functions $\Phi_n(\theta)$ are said to be linearly independent if no finite number of them are linearly dependent; that is, if no equation of the form

$$C_0\Phi_{r_0} + C_1\Phi_{r_1} + C_2\Phi_{r_2} + C_3\Phi_{r_3} + \cdots\cdots\cdots + C_n\Phi_{r_n} = 0, \qquad (11)$$

where the c's are constants not all zero, exists between a finite number of them. For the set of functions we are considering there certainly do exist such relations for some contours $r = R(\theta)$. For example, if $R = \sec\theta$ we have

$$\Phi_0 - \Phi_1 = 0; \qquad 2\Phi_2 - \Phi_4 = 0.$$

However, for no closed finite contour can such a relation exist. For, suppose we consider a contour $r = R(\theta)$ such that a linear relation of the form (11) exists. Then the left hand member of this equation is the value $V(R, \theta)$ of a harmonic function $V(r, \theta)$ on a finite closed contour; hence by Gauss's mean value theorem,

$$V(r, \theta) \equiv 0$$

for all points inside the contour, which is only possible if all the coefficients vanish. Hence the set of functions

$$\Phi_0 = 1,\ \Phi_1 = R\cos\theta,\ \Phi_2 = R\sin\theta,\ \Phi_3 = R^2\cos 2\theta,\ \cdots\cdots\cdots,$$

and in fact any infinite set $\Phi_n(\theta)$ obtained in the same way from a linearly independent set of two-dimensional harmonics $\phi_n(r, \theta)$, are linearly independent for *all closed contours*, in particular for polygons.

We can now apply Goursat's method and form a set, $\Psi_0, \Psi_1 \cdots\cdots$, of normal, orthogonal functions, as follows:

$$\begin{aligned}
\Psi_0 &= a_{00}\Phi_0 \\
\Psi_1 &= a_{10}\Phi_0 + a_{11}\Phi_1 \\
\Psi_2 &= a_{20}\Phi_0 + a_{21}\Phi_1 + a_{22}\Phi_2 \\
\Psi_3 &= a_{30}\Phi_0 + a_{31}\Phi_1 + a_{32}\Phi_2 + a_{33}\Phi_3 \qquad (12) \\
&\cdots\cdots\cdots\cdots\cdots \\
\Psi_m &= a_{m0}\Phi_0 + a_{m1}\Phi_1 + a_{m2}\Phi_2 + a_{m3}\Phi_3 + \cdots\cdots\cdots + a_{mm}\Phi_m \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Solving for the Φ's, we have

$$\begin{aligned}
\Phi_0 &= \quad 1 \quad = b_{00}\Psi_0 \\
\Phi_1 &= R\cos\theta = b_{10}\Psi_0 + b_{11}\Psi_1 \\
\Phi_2 &= R\sin\theta = b_{20}\Psi_0 + b_{21}\Psi_1 + b_{22}\Psi_2 \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\Phi_{2n-1} &= R^n\cos n\theta = b_{2n-1,\,0}\Psi_0 + b_{2n-1,\,1}\Psi_1 + b_{2n-1,\,2} + \cdots\cdots \\
&\qquad\qquad + b_{2n-1,\,2n-1}\Psi_{2n-1} \qquad (13) \\
\Phi_{2n} &= R^n\sin n\theta = b_{2n,\,0}\Psi_0 + b_{2n,\,1}\Psi_1 + b_{2n,\,2} + \cdots\cdots\cdots \\
&\qquad\qquad + b_{2n,\,2n-1}\Psi_{2n-1} + b_{2n,\,2n}\Psi_{2n} \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,
\end{aligned}$$

where

$$b_{nm} = (-1)^{n+m}\frac{1}{a_{m,\,m}\,a_{m+1,\,m+1}\cdots a_{nn}}\begin{vmatrix} a_{m+1,\,m} & a_{m+1,\,m+1} & 0 & \cdots & 0 \\ a_{m+2,\,m} & a_{m+2,\,m+1} & a_{m+2,\,m+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ & & & & 0 \\ a_{n-1,\,m} & a_{n-1,\,m+1} & a_{n-1,\,m+2} & \cdots & a_{n-1,\,n-1} \\ a_{n,\,m} & a_{n,\,m+1} & a_{n,\,m+2} & \cdots & a_{n,\,n-1} \end{vmatrix} \qquad (14)$$

Now expand $\log R(\theta)$ in terms of $\Psi_0, \Psi_1, \Psi_2, \cdots\cdots\cdots$ thus:

$$\log R(\theta) = \sum_{i=0}^{\infty} A_i\Psi_i,$$

where the coefficients are given by

$$A_i = \int_0^{2\pi} \log R(\theta)\ \Psi_i(\theta)\ d\theta. \qquad (15)$$

Substituting in equation (6), we have

$$\begin{aligned}
c_0\,(b_{00}\Psi_0) + c_1\,(b_{10}\Psi_0 + b_{11}\Psi_1) + c_2\,(b_{20}\Psi_0 + b_{21}\Psi_1 + b_{22}\Psi_2) \\
+ \cdots\cdots = \sum_{i=0}^{\infty} A_i\Psi_i, \qquad (16)
\end{aligned}$$

where $c_0, c_1, c_2, \cdots\cdots$ are the unknown constants required. Comparing coefficients of the Ψ's on each side of the equation, we have

$$\begin{aligned}
b_{00}\,c_0 + b_{10}\,c_1 + b_{20}\,c_2 + b_{30}\,c_3 + \cdots\cdots\cdots &= A_0 \\
b_{11}\,c_1 + b_{21}\,c_2 + b_{31}\,c_3 + \cdots\cdots\cdots &= A_1 \\
b_{22}\,c_2 + b_{33}\,c_3 + \cdots\cdots\cdots &= A_2 \qquad (17) \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &,
\end{aligned}$$

an infinite set of equations for the c's. Solving this set of equations formally for each c by eliminating the other c's one at a time, we have an expression

for each c in the form of an infinite series involving the b's and the A's. If, then, we substitute for the b's from equation (14), we arrive at the simple result

$$\begin{aligned}
c_0 &= a_{00}A_0 + a_{10}A_1 + a_{20}A_2 + a_{30}A_3 + a_{40}A_4 + \cdots\cdots\cdots \\
c_1 &= \quad a_{11}A_1 + a_{21}A_2 + a_{31}A_3 + a_{41}A_4 + \cdots\cdots\cdots \\
c_2 &= \quad\quad a_{22}A_2 + a_{32}A_3 + a_{42}A_4 + \cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_m &= \sum_{i=m}^{\infty} a_{im}A_i \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{18}$$

Substituting this solution in equation (6), we have

$$\sum_{i=0}^{\infty} a_{i0}A_i + \sum_{n=1}^{\infty}\left(\sum_{i=2n-1}^{\infty} a_{i,\,2n-1}A_iR^n\cos n\theta + \sum_{i=2n}^{\infty} a_{i,\,2n}A_iR^n\sin n\theta\right) = \sum_{i=0}^{\infty} A_i\Psi_i.$$

It is sufficient to multiply equations (12) by A_0, A_1, A_2, $\cdots\cdots$ respectively, and add, to verify formally the correctness of this equation. Now since we know the c's exist and are unique, if we can prove that equations (18) give an actual solution of (17), this is the only solution. To prove this we shall make use of the following theorem due to Anna J. Pell: *

Theorem: If the sequences $\{\lambda_k\}$ and $\{\mu_i\}$ are such that the sequence $\{b_{ik}/\lambda_k\}$ is of finite norm for every k, and the matrix $(\lambda_i a_{ki}/\mu_i)$ is limited, then for every sequence $\{A_k\}$ such that $\{\mu_k A_k\}$ is of finite norm, the system of equations

$$\sum_{i=k}^{\infty} b_{ik}c_i = A_k \qquad (k = 0,\ 1,\ 2,\ 3,\ \cdots\cdots)$$

has a solution c_k such that $\{\lambda_k c_k\}$ is of finite norm, and the solution is given by

$$c_k = \sum_{i=k}^{\infty} a_{ik}A_i \qquad (k = 0,\ 1,\ 2,\ 3,\ \cdots\cdots)$$

Making $\lambda_k = \mu_i = 1$, we can show that the three conditions required by the above theorem are satisfied, as follows:

(1) The first condition to be satisfied is that the sequence $\{b_{ik}\}$ shall be of finite norm for every k; that is, the coefficients in equations (13), taken by columns, must give sequences of finite norm. This we can prove as follows: Since the Ψ's are normal, orthogonal functions we have, for every n,

* Ann. of Math., Vol. 28, p. 35. We have changed Mrs. Pell's notation to conform to our own.

$$\int_0^{2\pi} R^{2n} \cos n\theta \, d\theta = b^2_{2n-1,\,0} + b^2_{2n-1,\,1} + \cdots\cdots + b^2_{2n-1,\,2n-1} \qquad (19)$$

$$\int_0^{2\pi} R^{2n} \sin n\theta \, d\theta = b^2_{2n,\,0} + b^2_{2n,\,1} + \cdots\cdots + b^2_{2n,\,2n}.$$

Hence for every k, we have

$$b^2_{2n-1,\,k} < \int_0^{2\pi} R^{2n} \cos n\theta \, d\theta < \int_0^{2\pi} R^{2n} \, d\theta = 2\pi R^{2n}_{max},$$

$$b^2_{2n,\,k} < \int_0^{2\pi} R^{2n} \sin n\theta \, d\theta < \int_0^{2\pi} R^{2n} \, d\theta = 2\pi R^{2n}_{max};$$

so that we have, for the norm of the sequence $\{b_{ik}\}$,

$$\sum_{i=k}^{\infty} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max}. \qquad k = 0,\ 1,\ 2,\ \cdots\cdots \qquad (20)$$

But the maximum value of $R(\theta)$ is always less than unity. Hence the right hand member of equation (20) is an absolutely convergent series. Thus we have proved that $\{b_{ik}\}$ is of finite norm for every k.

(2) The second condition is that the matrix (a_{ki}) of the coefficients in equations (18) shall be limited. This we can show to be true as follows: Consider first the matrix (b_{ik}) of equations (13). From (19) we have

$$\sum_{i=0}^{\infty} \sum_{k=0}^{i} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max} < M, \text{ a constant};$$

that is, the b's are of finite norm and hence * the matrix (b_{ik}) is limited. Now the matrix (a_{ik}) of equations (12) is the unique reciprocal of the matrix (b_{ik}), and hence is itself limited.† Therefore its conjugate, the matrix (a_{ki}) of equations (17) is limited.

(3) The third condition is evidently satisfied; for the sequence $\{A_i\}$ of the right hand members of equation (17) is of finite norm since the A's are the Fourier constants for the expansion of $\log R(\theta)$ in terms of the Ψ's.

Now by Mrs. Pell's theorem it follows that equations (18) give an actual solution of equation (17), and hence give us the required expressions for the constants c.

Formula (5) for the Green's function may now be written

$$G = -\log r + A_0\psi_0 + A_1\psi_1 + A_2\psi_2 + A_3\psi_3 + \cdots\cdots\cdots,$$

where

$$\psi_0 = a_{00}$$

$$\psi_1 = a_{10} + a_{11} r \cos\theta$$

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* Hellinger and Toeplitz: Math. Ann., Vol. 69, p. 307.

† Hellinger and Toeplitz: Math. Ann., *l. c.*, p. 311.

$$\psi_2 = a_{20} + a_{21} r \cos\theta + a_{22} r \sin\theta$$
$$\psi_3 = a_{30} + a_{31} r \cos\theta + a_{32} r \sin\theta + a_{33} r^2 \cos 2\theta$$
. . . . . .

Now substituting for the coefficients A_i from the equation (15), we have

$$G = -\log r + \sum_{i=0}^{\infty} \psi_i(r,\theta) \int_0^{2\pi} \log R(\theta)\, \Psi_i(\theta)\, d\theta, \tag{21}$$

which, since the matrix (a_{ik}) is limited, can be written

$$G = -\log r + \int_0^{2\pi} \log R(t) \sum_{i=0}^{\infty} [\psi_i(r,\theta)\, \Psi_i(t)]\, dt. \tag{22}$$

Thus we have an expression for the Green's function in terms of a *line integral around the boundary.* When $r = R(\theta)$, $\psi_i(r,\theta) = \Psi_i(\theta)$, and the last member of the equation (21) becomes the expansion for $\log R$ in terms of the Ψ's, so that G vanishes identically.

Application to a Square Contour.

As a simple application of the above, let us find the Green's function for a square with discontinuity at the center. If we make the apothem of the square unity and measure angles from it, we have for the function R

$$\begin{array}{lrl} R(\theta) = \sec\theta & -\pi/4 = & = \pi/4 \\ R(\theta) = \csc\theta & \pi/4 = & = 3\pi/4 \\ R(\theta) = -\sec\theta & 3\pi/4 = & = 5\pi/4 \\ R(\theta) = -\csc\theta & 5\pi/4 = & = 7\pi/4. \end{array} \tag{23}$$

Beginning with the first function $\Phi_0 = 1$ we shall first orthogonalize all the other functions to it by determining C_{2n-1} and C_{2n} to satisfy the following equations:

$$\int_0^{2\pi} (\Phi_{2n-1} - C_{2n-1}\Phi_0)\Phi_0\, d\theta = 0 \tag{24}$$
$$\int_0^{2\pi} (\Phi_{2n} - C_{2n}\Phi_0)\Phi_0\, d\theta = 0.$$

From the first of these equations we have $C_{2n-1} = 0$ for n odd and $n/2$ odd; that is,

$$C_1 = C_3 = C_5 = C_9 = C_{11} = C_{13} = C_{17} \cdots\cdots\cdots = 0.$$

For $n/2$ even

$$C_{2n-1} = 2/\pi \int_{\pi/4}^{\pi/4} \sec^n\theta \cos n\theta\, d\theta.$$

From the second of equations (22) we have $C_{2n} = 0$ for every n. Thus we obtain a new set of functions Φ', all of which are orthogonal to Φ_0, as follows:

$$\Phi'_1 = R\cos\theta \qquad \Phi'_2 = R\sin\theta$$
$$\Phi'_3 = R^2\cos 2\theta \qquad \Phi'_4 = R^2\sin 2\theta$$
$$\Phi'_5 = R^3\cos 3\theta \qquad \Phi'_6 = R^3\sin 3\theta$$
$$\Phi'_7 = R^4\cos 4\theta - .48826 \qquad \Phi'_8 = R^4\sin 4\theta$$
$$\Phi'_9 = R^5\cos 5\theta \qquad \Phi'_{10} = R^5\sin 5\theta$$
$$\cdots\cdots\cdots \qquad \cdots\cdots$$
$$\Phi'_{15} = R^8\cos 8\theta - 1.11259 \qquad \Phi'_{16} = R^8\sin 8\theta$$
$$\cdots\cdots\cdots \qquad \cdots\cdots$$
$$\Phi'_{23} = R^{12}\cos 12\theta - 3.01549 \qquad \Phi'_{24} = R^{12}\sin 12\theta$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Now we take Φ'_1 and orthogonalize Φ'_2, Φ'_3, Φ'_4, $\cdots\cdots$ to it in the same way as before. This process is continued until a sufficient number of orthogonal functions has been obtained. Finally these orthogonal functions must be normalized by dividing each function Φ by $\left[\int_0^{2\pi}\Phi^2\,d\theta\right]^{\frac{1}{2}}$. Denote by Ψ_0, Ψ_1, Ψ_2, $\cdots\cdots$, as in the preceding section, the normal, orthogonal functions obtained in this way.

Next we determine the A's by means of the formula

$$A_m = \int_0^{2\pi}\log R(\theta)\Psi_m(\theta)\,d\theta.$$

Finally, substituting in equations (18), we have the required constants for the square, namely,

$$c_0 = \sum_{i=0}^{\infty} a_{i0}A_i = -.23653 - .05516 + .00883 + .00474 + \cdots = -.28\cdots$$
$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$
$$c_7 = \sum_{i=7}^{\infty} a_{i7}A_i = \qquad -.11296 - .02100 - .00834 + \cdots = -.14\cdots$$
$$c_8 = c_9 = c_{10} = c_{11} = c_{12} = c_{13} = c_{14} = 0$$
$$c_{15} = \sum_{i=15}^{\infty} a_{i15}A_i = \qquad\qquad -.00794 - .00013 + \cdots = -.01\cdots$$
$$c_{16} = c_{17} = c_{18} = c_{19} = c_{20} = c_{21} = c_{22} = 0$$
$$c_{23} = \sum_{i=23}^{\infty} a_{i23}A_i = \qquad\qquad\qquad +.00153 + \cdots = +.002\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Putting these constants in equation (5), we have the following formula for

the Green's function for a square with the point of discontinuity at the center and diagonal of length 2:

$$G = -\log r - .28 - .14\, r^4 \cos 4\theta - .01\, r^8 \cos 8\theta + .002\, r^{12} \cos 12\theta + \cdots$$
$$= Re\,(-\log z - .28 - .14\, z^4 - .01\, z^8 + .002\, z^{12} + \cdots\cdots) \qquad (25)$$

approximately.

Thus we see that the expansion for u, that is, $G + \log r$ is the real part of an expansion in fourth powers of z. This may be verified by examining the expressions in elliptic functions, obtained by the method of images, for the square with the point of discontinuity and the origin of co-ordinates at the center. The character of the expansion for the Green's function for the square suggests immediately that for a regular polygon of p sides (p any positive integer) with discontinuity at the center, u will be the real part of an expansion in pth powers. It is evident at once that this is true when p becomes infinite, for in that case we have a circle for which u is constant. This may be shown to be true in general by making use of Schwarz' mapping function for a regular polygon of p sides.

On the Solution of Certain Types of Linear Differential Equations in Infinitely Many Variables.

By Webster G. Simon.

The main purpose of this paper is to prove the existence of certain types of solutions of particular kinds of linear differential equations with periodic coefficients in infinitely many variables. As a means to this end the existence of exponential solutions is established for certain types of linear differential equations in infinitely many variables with constant coefficients.

The starting point is the existence theorem given by von Koch,* and a generalization of Poincaré's theorem † concerning the development of the solutions of the differential equations as power series in a parameter μ when the functions appearing in the differential equations are themselves power series in μ. Then our work is very similar to that of the finite case, the finite determinants becoming infinite determinants which together with all their first minors converge absolutely.

The type of determinant used in this paper is more general than the normal determinant, and is the following.§ The determinant

$$\begin{vmatrix} 1+a_{11}, & a_{12}, & a_{13}, & \cdots \\ a_{21}, & 1+a_{22}, & a_{23}, & \cdots \\ a_{31}, & a_{32}, & 1+a_{33}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots \end{vmatrix}$$

is such that there exist two sets of positive constants $S_1, S_2, \cdots, T_1, T_2, \cdots$, which are of such a nature that $|a_{ij}| < S_i T_j$, and $\sum_{i=1}^{\infty} S_i T_i$ converges. This is a type of infinite determinant given by von Koch,§ which together with all its minors converges absolutely.

Using the methods thus indicated, we find that many of the phenomena of the finite systems are carried over into the infinite systems of differential equations.||

* Von Koch, Ofversigt af Kongliga Vetenskaps Akademiens Förhandlingar, Vol. 56 (1899), pp. 395-411.

† Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, chapter II.

§ Von Koch, *Acta Mathematica*, Vol. 24 (1901), pp. 89-122. Hereafter this paper will be denoted by K.

|| See Moulton and MacMillan, American Journal of Mathematics, Vol. 33 (1911), pp. 63-96.

I

The differential equations which we shall consider are of the general type

$$(1) \qquad x_i' = \sum_{j=1}^{\infty} \theta_{ij}(t) x_j, \quad i = 1, 2, \cdots \infty,$$

where x_i' is the derivative of x_i with respect to the independent variable t.

Concerning this system of differential equations we shall establish two existence theorems.

Theorem I. *Suppose the system* (1) *satisfies the following hypotheses:*
(H_1) *The* θ_{ij} (t) *are analytic functions of* t *when* $|t| < R$.
(H_2) *Positive constants,* $S_1, S_2, \cdots, T_1, T_2, \cdots$, *exist such that* $|\theta_{ij}(t)| < S_i T_j, |t| < R$, and $S_1T_1 + S_2T_2 + \cdots$ *converges.*
(H_3) *The* $x_i(0) = \beta_i$, *where the* β_i *are constants such that* $\beta_1T_1 + \beta_2T_2 \cdots$ *converges.*

Then there exists a unique system of functions

$$(2) \qquad x_i = \beta_i + \sum_{j=1}^{\infty} \beta_i^{(j)} t^j, \quad i = 1, 2, \cdots \infty,$$

which satisfy the system (1), *and which converge for* $|t| < R$.

Although this theorem was established by von Koch,* it will be proved briefly here because the notation and the results are essential for the later parts of this paper.

From (H_1) we have

$$(3) \qquad \theta_{ij}(t) = a_{ij}^{(0)} + a_{ij}^{(1)} t + \cdots, \quad i, j = 1, 2, \cdots \infty.$$

Upon substituting the series (2) in (1) and equating coefficients, we get

$$(4) \quad \begin{cases} \beta_i^{(1)} = \sum_{j=1}^{\infty} a_{ij}^{(0)} \beta_j^{(0)}, \\ 2\beta_i^{(2)} = \sum_{j=1}^{\infty} (a_{ij}^{(1)} \beta_j^{(0)} + a_{ij}^{(0)} \beta_j^{(1)}) \\ \cdots\cdots\cdots\cdots \\ (\lambda+1)\beta_i^{(\lambda+1)} = \sum_{j=1}^{\infty} \sum_{k=0}^{\lambda} a_{ij}^{(\lambda-k)} \beta_j^{(k)}, \text{ where } \beta_i^{(0)} = \beta_i. \end{cases}$$

We see that the formal solution is unique.

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* *Loc. cit.*

To prove the convergence of (2), consider the system of differential equations

$$(5) \qquad \xi_i' = S_i \sum_{j=1}^{\infty} T_j \xi_j, \quad i = 1, 2, \cdots \infty.$$

Upon substituting $\xi_i = \beta_i + \sum_{j=1}^{\infty} \gamma_i^{(j)} t^j$ in (5) and equating coefficients, we get

$$(6) \quad \begin{cases} \gamma_i^{(1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(0)}, \\ 2\gamma_i^{(2)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(1)}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (\lambda+1)\gamma_i^{(\lambda+1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(\lambda)}, \text{ where } \gamma_i^{(0)} = \beta_i. \end{cases}$$

On comparison of (6) and (4) we see that $\gamma_i^{(j)} > |\beta_i^{(j)}|$, $i, j = 1, 2, \cdots \infty$. Hence ξ_i dominates x_i for every i.

From (5) we have

$$(7) \qquad \frac{1}{S_1}\xi_1' = \frac{1}{S_2}\xi_2' = \cdots = \xi'.$$

On taking $\xi(0) = 0$, we have

$$(8) \qquad \xi_i - \beta_i = S_i \xi, \quad i = 1, 2, \cdots \infty.$$

Therefore each equation (5) reduces to

$$(9) \qquad \xi' = C\xi + K,$$

where $C = S_1 T_1 + S_2 T_2 + \cdots$, $K = \beta_1 T_1 + \beta_2 T_2 + \cdots$.
From the theory of a finite system of differential equations we know that (9) has a unique analytic solution for $|t| < R$. On combination of this fact with (8), we see that the solution (2) of (1) converges when $|t| < R$.

Now we define as a fundamental set of solutions of (1), a set such that every solution of (1) can be expressed as linear homogeneous functions with constant coefficients of the elements of the set. Then if we denote by ϕ_{ij} the elements of the fundamental set, it follows that the determinant of the ϕ_{ij} converges and is not zero for all $|t| < R$, and conversely. Furthermore it has the form $\Delta = \Delta_0 e^{\int \sum_{i=1}^{\infty} \theta_{ii}(t)\, dt}$, for all $|t| < R$, where Δ_0 is the value of the determinant Δ when $t = t_0$. If, for example, we take the solutions defined by $\phi_{ii}(0) = 1$ and $\phi_{ij}(0) = 0$, $i \neq j$, we see from (8) and (9) that

$$|\phi_{ii}| \leqq 1 + \frac{S_i T_i}{C} |(e^{Ct} - 1)|, \text{and } |\phi_{ij}| \leqq \frac{S_i T_j}{C} |(e^{Ct} - 1)|, i \neq j.$$

We see that the determinant of the ϕ_{ij} is absolutely convergent.

Theorem II. *Suppose that the system* (1) *satisfies the following hypotheses:*

(H_1) *The θ_{ij} are expansible as power series in a parameter μ, which converge for all real $|t| < R$ if $|\mu| < \rho$.*

(H_2) *For $\mu = 0$, the $\theta_{ij} = a_{ij}$, where the a_{ij} are constants.*

(H_3) *The θ_{ij} satisfy all the hypotheses of Theorem I uniformly with respect to μ if $|\mu| < \rho$.*

Then the solutions of (1) *can be expressed as power series in μ, which converge for $|\mu| \leqq \rho_0 < \rho$ and for all real $|t| < R$.*

In Theorem I we showed that $x_i - \beta_i = \sum_{j=1}^{\infty} \beta_i^{(j)} t^j$, $|t| < R, |\mu| < \rho$, where $n\beta_i^{(n)} = P_i^{(n)}(\beta_j^{(0)}, \cdots, \beta_j^{(n-1)})$.

In particular $\beta_i^{(1)} = \sum_{j=1}^{\infty} a_{ij}^{(0)} \beta_j^{(0)}$. The $a_{ij}^{(0)}$ are power series in μ. Hence $\beta_j^{(0)} a_{ij}^{(0)} = \sum_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$, $\beta_i^{(1)} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$, which converge for $|\mu| < \rho$. Then by a well-known theorem in the theory of double series, we infer that $\beta_i^{(1)} = \sum_{k=1}^{\infty} \beta_{ik}^{(1)} \mu^k$, $|\mu| < \rho$. In a similar fashion we proceed step by step, and show that the $\beta_i^{(n)}$ are also power series in μ, which converge for $|\mu| < \rho$. So we have

$$x_i = \sum_{k=1}^{\infty} \phi_{ik}(\mu), \quad i = 1, 2, \cdots \infty, \tag{10}$$

which is uniformly convergent for $|\mu| \leqq \rho_0 < \rho$, and where $\phi_{ik}(\mu) = \phi_{ik}^{(0)} + \phi_{ik}^{(1)} \mu + \cdots$. Then by a well-known theorem of Weierstrass, we know that we can rearrange the series (10), and write it in the form

$$x_i = \sum_{j=1}^{\infty} x_i^{(j)}(t) \mu^j, \quad |\mu| \leqq \rho_0 < \rho, \ |t| < R, \ i = 1, 2, \cdots \infty.$$

II

Now let us assume that the θ_{ij} of Theorem I are constants; and let these constants be denoted by a_{ij}. We inquire whether the system (1) has a solution of the form

$$x_i = c_i e^{at}, \tag{11}$$

where by (H_3) of Theorem I $\sum_{j=1}^{\infty} a_{ij} c_j$ converges. Necessary and sufficient con-

ditions that such a solution exist are

$$(12)\qquad \begin{cases} (a_{11}-a)c_1 + a_{12}c_2 + a_{13}c_3 + \cdots = 0, \\ a_{21}c_1 + (a_{22}-a)c_2 + a_{23}c_3 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

In order to see that (12) can have a solution for which the c_i are not all zero, we make the transformation $a = -\lambda^{-1}$, assuming of course that a is not zero. Then the equations (12) become

$$(13)\qquad \begin{cases} (1+\lambda a_{11})c_1 + \lambda a_{12}c_2 + \cdots = 0, \\ \lambda a_{21}c_1 + (1+\lambda a_{22})c_2 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

In order that the equations (13) have a solution in which the c_i are not all zero, it is necessary and sufficient that λ be a root of

$$(12)\qquad \Delta(\lambda) = \begin{vmatrix} 1+\lambda a_{11}, & \lambda a_{12}, & \cdots \\ \lambda a_{21}, & 1+\lambda a_{22}, & \cdots \\ \cdots, & \cdots, & \cdots \end{vmatrix} = 0,$$

Equation (14) is called the *characteristic equation.* Since the a_{ij} satisfy the hypotheses of Theorem I, it follows that $\Delta(\lambda)$ is an integral function * of λ.

If λ_0 is a root of (14), then there exists a finite number r such that $\Delta(\lambda_0)$ together with its minors of order $1, 2, \cdots, r-1$, vanishes; but there is at least one minor of order r which is different from zero.† Let the minor which is obtained by replacing the elements in the i_1th row and the k_1th column by unity and the remaining elements in that row and column by zero, and so on up to the i_rth row and the k_rth column be denoted by

$$\binom{i_1, i_2, \cdots, i_r}{k_1, k_2, \cdots, k_r}^{(\lambda_0)},$$

and suppose that this minor is one which is different from zero. Then the system (13) admits the solution ‡

$$(15)\qquad c_k = \frac{\binom{i_1, \cdots i_r}{k_1, \cdots k_r}^{(\lambda_0)} \xi_{k_1} + \binom{i_1, i_2 \cdots i_r}{k_1, k \cdots k_r}^{(\lambda_0)} \xi_{k_2} + \cdots + \binom{i_1, \cdots i_r}{k_1, \cdots k}^{(\lambda_0)} \xi_{k_r}}{\binom{i_1, \cdots i_r}{k_1, \cdots k_r}^{(\lambda_0)}}$$

depending on r independent parameters $\xi_{k_1}, \cdots \xi_{k_r}$, and admits no others.

To sum up our results we have the following

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* See K, p. 104. † See K, p. 108. ‡ See K, p. 109.

Theorem III: *If the* θ_{ij} *of* (1) *are constants and if* λ_0 *is a root of* $\Delta(\lambda) = 0$ *such that all minors of order* $r-1$, *but not all of order* r, *vanish for* $\lambda = \lambda_0$, *then there exist* r *independent solutions of* (1) *of the form* (11).

Now the characteristic equation, since it is an integral transcendental function, may have no finite roots, it may have a finite number of finite roots, or a denumerably infinite number of roots. In the first case the c_i are all zero, in the second case, a finite number of solutions of the form (11) exists and in the last case an infinite number of the form (11) exists. This is the case we shall study, for this infinite set constitutes a fundamental set of solutions as the following discussion shows.

For each λ that is a root of $\Delta(\lambda) = 0$ there is a solution of (1) of the form

$$x_{ij} = c_{ij} e^{a_j t}, \quad i, j = 1, 2, \cdots \infty. \tag{16}$$

We know that the system (1) admits the fundamental set of solutions ϕ_{ij}, $i, j = 1, 2, \cdots \infty$, where $\phi_{ii}(0) = 1$, $\phi_{ij}(0) = 0$, $i \neq j$. Consequently

$$x_{i_1} = \sum_{k=1}^{\infty} A_k^{(1)} \phi_{ik}, \quad i = 1, 2, \cdots \infty.$$

First let us assume that the a_j are all distinct. Then it follows immediately that the system (16) constitutes a fundamental set. For consider x_{i_1}, ϕ_{ij}, $i = 1, 2, \cdots \infty$, $j = 2, 3, \cdots \infty$.

$$\begin{vmatrix} \sum_{k=1}^{\infty} A_k^{(1)} \phi_{1k}, & \phi_{12}, & \cdots \\ \sum_{k=1}^{\infty} A_k^{(1)} \phi_{2k}, & \phi_{22}, & \cdots \\ \cdots\cdots, & \cdots, & \cdots \end{vmatrix} = A_1^{(1)} \begin{vmatrix} \phi_{11}, & \phi_{12}, & \cdots \\ \phi_{21}, & \phi_{22}, & \cdots \\ \cdots, & \cdots, & \cdots \end{vmatrix}$$

Choose the initial conditions such that $A_1^{(1)} = 1$, which is no actual restriction since we are merely determining the arbitrary constant which enters, and we can multiply our solution by a constant, not zero, when we are through if we choose. Thus x_{i_1}, ϕ_{ij}, $i = 1, 2, \cdots \infty$, $j = 2, 3, \cdots \infty$, constitute a fundamental set. Similarly we show that x_{i_1}, x_{i_2}, ϕ_{ij}, constitute a fundamental set, where $j = 3, 4, \cdots \infty$. Continuing in this manner and passing to the limit we see that the x_{ij}, $i, j = 1, 2, \cdots \infty$, constitute a fundamental set. A result of this determination of the arbitrary constant is that $c_{ii} = 1$.

Next let $a_2 = a_1$, and $a_j \neq a_1$, $j = 3, 4, \cdots \infty$. There are two cases that may arise here, viz., all of the first minors of $\Delta(\lambda)$ vanish when $a = a_1$, or not all vanish. Consider the former case. Then (15) assures us that we can solve for the c_{ij} in terms of two of them. On choosing the notation so

that $\binom{1\ 2}{1\ 2}$ is not zero, we get two independent solutions of the form

$$x_{i1} = c_{i1}e^{a_1 t},\ x_{i2} = c_{i2}e^{a_1 t},\ i = 1, 2, \cdots \infty,$$

which with ϕ_{ij}, $j = 3, 4, \cdots \infty$, constitute a fundamental set; for

$$\begin{vmatrix} x_{11}, & x_{12}, & \phi_{13}, & \cdots \\ x_{21}, & x_{22}, & \phi_{23}, & \cdots \\ \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdots \end{vmatrix} = \begin{vmatrix} A_1^{(1)}\phi_{11} + A_2^{(1)}\phi_{12}, & A_1^{(2)}\phi_{11} + A_2^{(2)}\phi_{12}, & \phi_{13}, & \cdots \\ A_1^{(1)}\phi_{21} + A_2^{(1)}\phi_{22}, & A_1^{(2)}\phi_{21} + A_2^{(2)}\phi_{22}, & \phi_{23}, & \cdots \\ \cdots\cdots\cdots, & \cdots\cdots\cdots, & \cdot\cdot, & \cdots \end{vmatrix}$$

and the $A_2^{(1)}$, $A_1^{(2)}$ may be taken to be zero, and $A_1^{(1)} = A_2^{(2)} = 1$. So these solutions, we see, constitute a fundamental set.

Next let us assume that not all the first minors of $\Delta(\lambda)$ vanish when $a = a_2 = a_i$; and let us choose the notation so that the first minor $\binom{1}{1}$ is not zero . Then to get a solution associated with a_1, we make the transformation $y = b_1x_1 + b_2x_2 + \cdots$, $y' = b_1x'_1 + b_2x'_2 + \cdots$ and, if possible, determine the b's in such a manner that $y' = ay$. On substituting these equations in (1) we get $b_1[a_{11}x_1 + a_{12}x_2 + \cdots] + b_2[a_{21}x_1 + a_{22}x_2 + \cdots] + \cdots = a[b_1x_1 + b_2x_1 + \cdots]$.

This equation must hold for all initial values of the x's. Therefore it is an identity in them; and the b's satisfy the following set of equations

$$(17) \qquad \begin{cases} (a_{11} - a)b_1 + a_{21}b_2 + \cdots = 0, \\ a_{12}b_1 + (a_{22} - a)b_2 + \cdots = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot. \end{cases}$$

In order that these equations have a solution for the b's not all zero, we get as a necessary and sufficient condition, just as in the case of c's in (13), $\Delta(\lambda) = 0$. This condition is satisfied, for we have assumed that when $a = a_1$, the characteristic equation is satisfied. Furthermore we have chosen the notation such that the minor $\binom{1}{1}$ is not zero when $a = a_1$. Then we get for the b's

$$b_j = m_jb_1, \quad j = 1, 2, \cdots \infty,$$

where $m_j = \frac{\binom{1}{j}}{\binom{1}{1}}$. For convenience we shall take $b_1 = 1$. Then the equations become

$$(1') \begin{cases} y'_1 = a_1y_1, \\ x'_2 = a_{21}y_1 + (a_{22} - a_{21}m_2)x_2 + (a_{23} - a_{21}m_3)x_3 + \cdots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \\ x'_n = a_{n1}y_1 + (a_{n2} - a_{n1}m_2)x_2 + (a_{n3} - a_{n1}m_3)x_3 + \cdots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot. \end{cases}$$

Next we make the transformation $y_2 = d_1 y_1 + d_2 x_2 + \cdots$, $y'_2 = d_1 y'_1 + d_2 x'_2 + \cdots$ and determine, if possible, $d_2, d_3, \cdots$, such that $y'_2 = a y_2 + y_1$. Then, as in the foregoing, we get the following, infinite set of equations for the determination of the d's

$$(17') \begin{cases} (a_1 - a) d_1 + a_{21} d_2 + a_{31} d_3 + \cdots = 1, \\ 0 + (a_{22} - a_{21} m_2 - a) d_2 + \cdots = 0, \\ \cdots\cdots\cdots\

Then by a precisely similar argument to the foregoing we can show that

$$x_{i_1}, x_{i_2}, \phi_{ij}, \ i = 1, 2, \cdots \infty, \ j = 3, 4, \cdots \infty,$$

constitute a fundamental set.

When for a given value of a the characteristic equation has a root of higher multiplicity, discussions similar to the foregoing must be made.

We should expect to treat next the case that $\Delta(\lambda) = 0$ has a root of infinite multiplicity. But such a case cannot arise. For we have seen that $\Delta(\lambda)$ is an integral transcendental function of λ. Therefore it can be expanded in a Taylor's series in the neighborhood of any finite λ. Suppose that λ_0 were a finite root of infinite multiplicity. Then $\Delta(\lambda)$ vanishes together with all of its derivatives when $\lambda = \lambda_0$. Hence $\Delta(\lambda)$ vanishes at every point in the neighborhood of λ_0. Then it is identically zero. But this is not true, for when $\lambda = 0$, $\Delta(\lambda) = 1$. Hence $\Delta(\lambda)$ has no root of infinite multiplicity.

If $a = 0$, we can not make the transformation which carries the system (12) into the system (13), and the determinant of (12) diverges. Then we do not know whether the equations (12) have a solution for the c_j not all zero; but each special case must be considered as it arises.

To sum up our results we have

Theorem IV. *If and only if* $\Delta(\lambda)$ *has an infinite number of roots* λ_i, *the system* (1) *has a fundamental set of solutions each of whose elements is of the form*

$$x_{ij} = e^{a_j t}\psi_{ij}(t), \ i, j = 1, 2, \cdots \infty,$$

where $a_1 = -1/\lambda_i$ *and the* ψ_{ij} *are polynomials in* t *of degree at most* $(n-1)$, *and* n *is the order of multiplicity of the root* λ_j.

III

Now we shall assume that the θ_{ij} of Theorem I are periodic with the period 2π. We have seen that the system (1) has, as a fundamental set of solutions, the set $\phi_{ij}(t)$, where

$$\phi_{ii}(0) = 1, \ \phi_{ij}(0) = 0, \ i \neq j, \ i, j, = 1, 2, \cdots \infty. \tag{18}$$

Let us make the transformation

$$x_i = e^{at} y_i, \tag{19}$$

where a is an undetermined constant. Then the equations (1) become

$$y_i' + a y_i = \sum_{j=1}^{\infty} \theta_{ij}(t) y_j, \ i = 1, 2, \cdots \infty. \tag{20}$$

Since the system ϕ_{ij} constitutes a fundamental set of solutions of (1), any solution of (20) can be written

$$(21) \qquad y_i = e^{-at}\sum_{j=1}^{\infty} A_j\phi_{ij}(t),\ i=1,\ 2,\ \cdots\ \infty.$$

We now inquire whether it is possible to determine the A_j and a so that the y_i defined by (21) shall be periodic with the period 2π. From the form of (20) it is evident that necessary and sufficient conditions that the y_i be periodic with the period 2π are

$$(22) \qquad y_i(2\pi) - y_i(0) = 0,\ i=1,2,\ \cdots\ \infty.$$

On imposing these conditions on (21), we get

$$(23) \qquad \sum_{j=1}^{\infty} A_j[e^{-2a\pi}\phi_{ij}(2\pi) - \phi_{ij}(0)] = 0,\ i=1,2,\ \cdots\ \infty.$$

Then we make the transformation

$$(24) \qquad e^{-2a\pi} = \sigma/\sigma - 1,$$

assuming of course that $e^{-2a\pi} \neq 1$. Then the equations (23) become

$$(25) \qquad \sum_{j=1}^{\infty}{}' A_j\sigma\phi_{ij}(2\pi) + A_i[1+\sigma(\phi_{ii}(2\pi) - 1] = 0, \qquad i \neq j,\ i=1,2,\ \cdots\ \infty.$$

In order that these equations have a solution other than that in which the A_j are all zero, it is necessary that the determinant of the coefficients be zero. On writing the $\phi_{ij}(2\pi)$ simply ϕ_{ij}, the determinant is

$$(26) \qquad D(\sigma) = \begin{vmatrix} 1+\sigma(\phi_{11}-1), & \sigma\phi_{12}, & \cdots \\ \sigma\phi_{21}, & 1+\sigma(\phi_{22}-1), & \cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots \end{vmatrix} = 0.$$

This determinant, we see from the fact that the determinant of ϕ_{ij} converges absolutely, is an integral function of σ. Equation (26) is called the *fundamental equation* associated with the period 2π, and does not admit $\sigma = 0$ as a root.

As in the characteristic equation in II, the fundamental equation may have no finite root, a finite number of finite roots, or a denumerably infinite number of finite roots.

As in the case of constant coefficients, if the fundamental equation has no roots, the system (1) has no solution of the form (19), if the fundamental equation has a finite number of finite roots, the system (1) has a finite number of solutions of the form (19), and finally if the fundamental equation has an

infinite number of finite roots, the system (1) has a fundamental set of solutions, each of the elements of which is of the form (19).

The discussion of the form of the solutions follows precisely the same lines as that in the case of constant coefficients except when σ_0 is an n-fold root of (26) and not all the $(n-k)$th, $k>0$, minors of $D(\sigma)$ vanish for $\sigma=\sigma_0$.

Now let us assume that $\sigma=\sigma_0$ is an n-fold root of (26) and not all the first minors of $D(\sigma)$ vanish. Then there is only one solution of (1) of the form $x_i=e^{a_1 t}y_i$, where the y_i are expressed as in (21). And let us choose the notation so that a minor corresponding to the elements of the first column is not zero.

Then we take as a new set of solutions

$$(27)\qquad x_{i_1}=e^{a_1 t}y_{i_1},\ x_{ij}=\phi_{ij}(t),\ i=1,2,\cdots\infty,\ j=2,3,\cdots\infty.$$

This set of solutions can be shown to constitute a fundamental set in a precisely similar fashion to that in II. Then we make the transformation

$$x_i=e^{at}z_i,\ i=1,2,\cdots\infty.$$

As above we get

$$(28)\qquad z_i=e^{-at}[A_1e^{a_1 t}y_{i_1}+\sum_{j=2}^{\infty}A_j\phi_{ij}(t)],\ i=1,2,\cdots\infty.$$

Necessary and sufficient conditions that the z_i be periodic with the period 2π are

$$z_i(2\pi)-z_i(0)=0,\ i=1,2,\cdots\infty.$$

On imposing these conditions on (28). we get

$$(29)\qquad A_1[e^{-2(a-a_1)\pi}y_{i_1}(2\pi)-y_{i_1}(0)]+\sum_{j=2}^{\infty}A_j[\phi_{ij}(2\pi)e^{-2a\pi}-\phi_{ij}(0)]=0.$$

After setting $e^{-2a\pi}=\sigma/\sigma-1$, the equations (29) become

$$(30)\qquad A_1(1-\sigma/\sigma_1)y_{i_1}(0)+\sum_{j=2}^{\infty}{}'A_j\sigma\phi_{ij}+A_k[1+\sigma(\phi_{kk}-1)]=0,$$
$$k\neq j,\quad k=2,3,\cdots\infty,\quad i=1,2,\cdots\infty.$$

The fundamental equation for the equations (30) is

$$(31)\qquad \begin{vmatrix}(1-\sigma/\sigma_1)y_{11}(0), & \sigma\phi_{12}, & \cdots\\ (1-\sigma/\sigma_1)y_{21}(0), & 1+\sigma(\phi_{22}-1), & \cdots\\ \cdot & \cdot & \cdot\end{vmatrix}=0.$$

Making use of (21) and taking $A_1=1$, equation (31) becomes

$$\begin{vmatrix} \sigma\sum_{j=1}^{\infty} A_j\phi_{1j} - (\sigma-1), & \sigma\phi_{12} & \cdots \\ \sigma\sum_{j=1}^{\infty} A_j\phi_{2j} - A_2(\sigma-1), & 1+\sigma(\phi_{22}-1), & \cdots \\ \cdots\cdots\cdots, & \cdots\cdots\cdots, & \cdots \end{vmatrix} = D(\sigma) = 0.$$

So we have

$$(32)\quad D(\sigma) = (1-\sigma/\sigma_1)\,D_1 = (1-\sigma/\sigma_1)\begin{vmatrix} y_{11}(0), & \sigma\phi_{12}, & \cdots \\ y_{21}(0), & 1+\sigma(\phi_{22}-1), & \cdots \\ \cdots\cdots\cdots, & \cdots\cdots\cdots, & \cdots \end{vmatrix} = 0.$$

Since $\sigma = \sigma_1$ is an n-fold root of $D(\sigma) = 0$, $D_1(\sigma)$ has the factor $(1-\sigma/\sigma_1)^{n-1}$. Since (27) constitutes a fundamental set, any solution can be expressed in the form

$$(33)\qquad x_i = B_1 e^{a_1 t} y_{i1} + \sum_{j=2}^{\infty} B_j\phi_{ij},\ i = 1, 2, \cdots, \infty.$$

Now let us make the transformation, corresponding to (19), to get a second solution associated with a_1.

$$(34)\qquad x_{i2} = e^{a_1 t}(y_{i2} + t y_{i1}),\ i = 1, 2, \cdots \infty.$$

On imposing the condition that x_{i2} shall satisfy the system (1), we find since $e^{a_1 t} y_{i1}$ is a solution,

$$(35)\qquad y'_{i2} + a_1 y_{i2} = \sum_{j=2}^{\infty} \theta_{ij}(t)\, y_{j2} - y_{i1},\ \ i = 1, 2, \cdots \infty.$$

From the form of (35) we see that sufficient conditions that the y_{i2} shall be periodic with the period 2π are

$$(36)\qquad y_{i2}(2\pi) - y_{i2}(0) = 0 = \sum_{j=2}^{\infty} B_j[e^{-2a_1\pi}\phi_{ij}(2\pi) - \phi_{ij}(0)] - 2\pi y_{i1}(0) = 0.$$

On substituting $e^{-2a_1\pi} = \sigma_1/\sigma_1 - 1$, (36) becomes

$$(37)\qquad -2\pi(\sigma_1 - 1)y_{i1}(0) + \sum_{j=2}^{\infty}{}' B_j\phi_{ij} + B_k[1 + \sigma_1(\phi_{kk} - 1] = 0,\ \ k \neq j.$$

The condition that these equations be consistent is, since $\sigma_1 \neq 1$,

$$D_1(\sigma) = \begin{vmatrix} y_{11}(0), & \sigma_1\phi_{12}, & \cdots \\ y_{21}(0), & 1+\sigma_1(\phi_{22}-1), & \cdots \\ \cdots, & \cdots\cdots, & \cdots \end{vmatrix} = 0.$$

In (32) we showed that D_1 vanishes $(n-1)$ times when $\sigma = \sigma_1$. By hypotheses not all the first minors corresponding to the elements of the first column

are zero for $\sigma = \sigma_1$. Hence we can solve equations (37) for $B_2, B_3, \cdots$, terms of $y_{i_1}(0)$. Consequently in this case we get a second solution associated with a_1, which is of the form (34).

In a similar manner we can go ahead step by step and get the following group of solutions associated with a_1

$$\begin{cases} x_{i_1} = e^{a_1 t} y_{i_1}, \\ x_{i_2} = e^{a_1 t}[y_{i_2} + t y_{i_1}], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{in} = e^{a_1 t}[y_{in} + t y_{i n-1} + \cdots + 1/(n-1)!\ t^{n-1} y_{i_1}]. \end{cases}$$

If σ_1 is a triple root of $D(\sigma) = 0$, such that all first minors are zero, but not all the second minors are zero, the solutions associated with a_1 are

$$x_{i_1} = e^{a_1 t} y_{i_1},\ x_{i_2} = e^{a_1 t} y_{i_2},\ x_{i_3} = e^{a_1 t}[y_{i_3} + t(y_{i_1} + y_{i_2})].$$

All the sub-cases can be treated, as they arise, by the methods given here.

IV

Now let us assume that the system of differential equations (1) satisfy the hypotheses of the second existence theorem, and in addition the coefficients in the power series expansions of the θ_{ij} are separately periodic with the period 2π. That theorem tells us that the solutions of (1) can be written in the form

$$(39) \qquad x_{ij} = \sum_{k=1}^{\infty} x_{ij}^{(k)}(t)\mu^k,\ i, j = 1, 2, \cdots \infty.$$

And we shall take the initial conditions such that

$$(40) \qquad x_{ij}(0) = \sum_{k=1}^{\infty} x_{ij}^{(k)}(0)\mu^k \equiv c_{ij},$$

whence

$$x_{ij}^{(0)}(0) = c_{ij},\ x_{ij}^{(k)}(0) = 0,\ k = 1, 2, \cdots \infty.$$

where the c_{ij} are constants such that $c_{ii} = 1$, and their determinant is absolutely convergent. These conditions coupled with the fact that if the determinant of a set of solutions converges and is not zero when $t = 0$, it converges and is not zero for every value of t for which $\sum_{i=1}^{\infty} \theta_{ii}$ converges, show that the system (40) constitutes a fundamental set of solutions.

Now we inquire if we can find solutions of (1) of the form

$$(41) \qquad x_i = e^{at} y_i,$$

where the y_i are periodic with the period 2π, and a is a constant which remains

to be determined. After making the transformation (41), the differential equations and their solutions become

$$(42)\begin{cases} y'_i + ay_i = \sum_{j=1}^{\infty}[a_{ij} + \sum_{k=1}^{\infty}\theta_{ij}^{(k)}\mu^k]y_j, \ i = 1, 2, \cdots \infty, \\ y_i \qquad = \sum_{j=1}^{\infty} A_j e^{-at}[x_{ij}^{(0)}(t) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t)\mu^k], \ i = 1, 2, \cdots \infty. \end{cases}$$

On imposing the conditions that the y_i be periodic with the period 2π, viz., $y_i(2\pi) - y_i(0) = 0$, we get

$$(43) \qquad 0 = \sum_{j=1}^{\infty} A_j[e^{-2a\pi}x_{ij}^{(0)}(2\pi) - c_{ij} + e^{-2a\pi}\sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi)\mu^k], \ i = 1, 2, \cdots \infty.$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the equations (43) become

$$(44) \qquad 0 = \sum_{j=1}^{\infty} A_j[\sigma\{x_{ij}^{(0)}(2\pi + \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi)\mu^k\} - c_{ij}(\sigma - 1)] = 0, \quad i = 1, 2, \cdots \infty.$$

To avoid the trivial case where the A_j are all zero, we must set the determinant

$$(45) \qquad D(\sigma, \mu) = |\ [\sigma\{x_{ij}^{(0)}(2\pi) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(\pi)\mu^k\} - c_{ij}(\sigma - 1)]\ | = 0,$$

which is a condition on the undetermined constant a. Since the $x_{ij}^{(0)} + \sum_{k=1}^{\infty} x_{ij}^{(k)}\mu^k$ are the elements of a fundamental set, and $c_{ii} = 1$, it follows that $D(\sigma, \mu)$ converges absolutely for all finite σ's. If the fundamental equation is satisfied when $a = a_0$, it is also satisfied when $a = a_0 + \nu\sqrt{-1}$, where ν is any integer; but all distinct solutions of the differential equations can be obtained by taking $\nu = 0$, since the ratios of the A_j are the same for $\nu = 0$ as for $\nu = p$. And for $\mu = 0$ the equation (45) reduces to

$$(46) \qquad D_0 = |\ \sigma x_{ij}^{(0)}(2\pi) - c_{ij}(\sigma - 1)\ | = 0.$$

When $\mu = 0$, we have the case of constant coefficients which we treated in II. Now we shall determine the c_{ij} and the $a^{(0)} = \lambda^{-1}$ of the solutions as we did there. But having the c_{ij} we might also determine the a of the solutions by means of (46). Since for given initial conditions the solution of the differential equations is unique, and since the initial conditions are the same in the two cases, the solutions obtained by the former method and the solution obtained by means of (46) are the same.

We recall that the c_{ij} are not all zero if and only if the characteristic equation has finite roots. Therefore for every root of the characteristic equation there is a corresponding root σ of (46); and conversely, for every root of (46) there is as corresponding root of the characteristic equation. There-

fore if the characteristic equation has no finite roots, then (46) has no finite roots, if the former has only a finite number of finite roots, then the latter has only a finite number of finite roots, then (46) has an infinite number of finite roots which yield independent solutions of the system (1).

Now since the solution (41) converges for all μ sufficiently small, including zero, the failure to find an a for $\mu = 0$ is not due to clumsy analytic methods, but shows that the system (1) has no solution of the form (41). From the theory of implicit functions we know that if (46) is satisfied when $a = a_0$, then we can solve for a as power series in μ, provided μ is sufficiently small. Therefore the existence of a root of the characteristic equation is both necessary and sufficient for the existence of a solution of the system (1) of the form (41), and the existence of an infinite number of roots of the characteristic equation is necessary and sufficient for the existence of a fundamental set of solutions of (1), each of the elements of which is of the form (41).

We shall study in detail the case that the characteristic equation has an infinite number of roots; and we shall use only those values of a obtained from (45) which for $\mu = 0$ reduce to the values of a obtained from the characteristic equation.

When $e^{-2a\pi} = 1$, the transformation $e^{-2a\pi} = \sigma/\sigma - 1$ can not be made. Then as in the case of constant coefficients when $a = 0$, we can make no general statement about the solution, as the determinant of the A_i diverges then.

V

Solutions when the $a_i^{(0)}$ are distinct and $a_i^{(0)} - a_j^{(0)} \not\equiv 0 \bmod \sqrt{-1}$.

The part of (45) which is independent of μ is

$$D_0 = | \sigma c_{ij} e^{2a_j^{(0)}\pi} - c_{ij}(\sigma - 1) | = | c_{ij} | \prod_{j=1}^{\infty} \left(1 + \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{-2a\pi}} e^{-2a\pi}\right)$$

If (45) were an identity in μ, its roots would be the roots of (46), viz., $a = a_j^{(0)}$. Let us assume that we have the general case in which it is not an identity in μ, and set

$$a = a_k^{(0)} + \delta_k. \tag{47}$$

Then we get

$$D(\sigma, \mu) = D_0 + \mu F_k(\delta_k, \mu) = \tag{48}$$

$$| c_{ij} | \left(1 - \frac{e^{-2\delta_k\pi} - e^{-2(a^{(0)}{}_k + \delta_k)\pi}}{1 - e^{-2(a^{(0)}{}_k + \delta_k)\pi}}\right) \prod_{j=1}^{\infty}{}' \left(1 + \frac{1 - e^{-2a_j^{(0)}\pi}}{1 - e^{-2(a^{(0)}{}_k + \delta_k)\pi}} e^{-2(a_k^{(0)} + \delta_k)\pi}\right)$$
$$+ \mu F_k(\delta_k, \mu) \quad j \neq k.$$

where $F_k(\delta_k, \mu)$ is a power series in μ and δ_k, converging for

$$|\delta_k| < \infty, |\mu| < \rho > 0.$$

Since by hypothesis no two of the $a_1^{(0)}$ differ by an imaginary integer, the expansion of (48) as a power series in δ_k and μ contains a term in δ_k of the first degree and no term independent of both μ and δ_k. Therefore we know by the theory of implicit functions that (48) can be solved uniquely for δ_k as a power series of the form

$$\delta_k = \mu P_k(\mu), \tag{49}$$

which converges for $|\mu| > 0$ but sufficiently small.

Now we substitute this value of $a = a_k^{(0)} + \delta_k$ in (44), and get an infinite number of linear homogeneous equations for the A_j whose determinant converges and is zero, but the first minors of that determinant are not all zero, since by hypothesis, the roots of $D_0 = 0$ are all distinct and no two differ by an imaginary integer. Consequently these equations determine uniquely the ratios of the A_j as power series in μ, which converge for μ sufficiently small. On substituting these ratios in (42) we have the particular solution y_{ik}, $i = 1, 2, \cdots \infty$, expanded as a power series in μ. Hence we may write it

$$y_{ik} = \sum_{j=1}^{\infty} y_{ik}^{(j)}(t)\mu^j. \tag{50}$$

Since the periodicity conditions have been satisfied,

$$y_{ik}(t + 2\pi) - y_{ik}(t) = \sum_{j=1}^{\infty} [y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t)]\mu^j = 0,$$

for all μ sufficiently small and for all real t. Therefore

$$y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t) = 0, \ j = 0, 1, 2, \cdots \infty,$$

whence it follows that the $y_{ik}^{(j)}$, $j = 0, 1, 2, \cdots \infty$, are separately periodic.

A solution is found in a similar fashion for each $a_j^{(0)}$.

VI

Solutions when no two of the $a_j^{(0)}$ are equal but when $a_2^{(0)} - a_1^{(0)} \equiv 0 \bmod \sqrt{-1}$.

Suppose that when $\mu = 0$ the characteristic equation has two roots such that $a_2^{(0)}$ and $a_1^{(0)}$ differ by an imaginary integer and that none of the other $a_j^{(0)}$ are congruent to $a_1^{(0)}$ mod $\sqrt{-1}$. Then we see from (45)

$$(51)\qquad D(\sigma,\mu)=$$
$$|\,c_{ij}\,|\left[1-\frac{1-e^{2a_1^{(0)}\pi}}{1-e^{2(a_1^{(0)}+\delta_1)\pi}}\right]^2\prod_{j=3}^{\infty}\left[1-\frac{1-e^{2a_j^{(0)}\pi}}{1-e^{2(a_1^{(0)}+\delta_1)\pi}}\right]+\mu F(\mu,\delta,)=0,$$

where as in (47) we have set $a=a_1^{(0)}+\delta_1$. The term of lowest degree in δ_1 alone is found by expanding the first bracket and turns out to be of the second degree. To get the terms in μ alone we suppress those involving δ_1, after which we get a factor μ from each of the first two columns. So we see that in general the term of lowest degree in μ alone will be in this case of the second degree. Hence we have

$$(52)\qquad D=|\,c_{ij}\,|\left[1-\frac{1-e^{2a_1^{(0)}\pi}}{1-e^{2(a_1^{(0)}+\delta_1)\pi}}\right]^2\prod_{j=3}^{\infty}\left[1-\frac{1-e^{2a_j^{(0)}\pi}}{1-e^{2(a_1^{(0)}+\delta_1)\pi}}\right]$$
$$+\delta_1\mu F_1(\delta_1,\mu)+\delta^2F_2(\delta_1,\mu)=0.$$

In a similar manner if p of the $a_j^{(0)}$ are congruent to $a_1^{(0)}$ mod $\sqrt{-1}$, then the term of lowest degree in δ_1 alone is of degree p, and in μ alone it is of at least the pth degree.

The problem of the form of the solution of (52) is one of implicit functions. Writing the first terms explicitly we have

$$\delta_1^2+\kappa_{11}\delta_1\mu+\kappa_{02}\mu_2+\text{terms of higher degree}=0,$$

where $\kappa_{11}, \kappa_{02}, \cdots$, are constants independent of δ_1 and μ. On factoring the quadratic terms we get

$$(53)\qquad (\delta_1-d_1\mu)\,(\delta_1-d_2\mu)+\text{terms of higher degree}=0.$$

If d_1 and d_2 are distinct, there are two solutions, and these have the form *

$$(54)\qquad \delta_{11}=d_1\mu+\mu^2P_1(\mu),\ \delta_{12}=d_2\mu+\mu^2P_2(\mu),$$

where P_1 and P_2 are power series which converge for μ sufficiently small. In this case the solutions are found as in V.

But if d_1 and d_2 are equal, the character of the solution is in general quite different and depends upon the terms of higher degree than the second. In general it will be a power series in $\pm\sqrt{\mu}$. This case we shall consider in detail.

We see from the form of (53) that the expansion of a_1 as a power series in $\sqrt{\mu}$ will contain no term in $\sqrt{\mu}$ to the first power, but will have the form

$$a_1=a_1^{(0)}+0\mu^{1/2}+a_1^{(1)}\mu+a_1^{(3/2)}\mu^{(3/2)}+\cdots\cdots.$$

Suppose that this expansion has been obtained from equation (45).

* Chrystal, "Algebra," Vol. 2, pp. 358 ff.

Then since a_1 is not a multiple zero of D, not all the first minors of D are zero when $a = a_1$. The ratios of the A_j will be determined from (44). If $\mu\Delta$ is a non-vanishing first minor corresponding to an element in the first column of D, it follows from the form of (45), remembering that we have $x_{ij}^{(0)}(t) = c_{ij}e^{a_j^{(0)}t}$, that solving (44), we get

$$A_2 = \frac{\mu\Delta_2}{\mu\Delta} A_1,\ A_j = \frac{\mu^2\Delta_j}{\mu\Delta} A_1,\ j = 3,\ 4,\ \cdots,\ \infty,$$

where $\Delta = \Delta^{(0)} + \Delta^{(1/2)}\mu^{1/2} + \Delta^{(1)}\mu + \cdots\cdots,$

$\Delta_j = \Delta_j^{(0)} + \Delta_j^{(1/2)}\mu^{1/2} + \Delta_j^{(1)}\mu + \cdots\cdots.$

On substituting these series for the A_j in (42) we find that the y_{i1} are developable as series of the form

$$y_{i1} = y_{i1}^{(0)} + y_{i1}^{(1/2)}\mu^{1/2} + y_{i1}^{(1)}\mu + \cdots\cdots \quad i = 1,\ 2,\ \cdots\ \infty.$$

So we see that in general the y_{i1} carry terms in $\sqrt{\mu}$ although the term in $\sqrt{\mu}$ is absent in the expansion for a_1.

However, if all the first minors corresponding to the elements of the first column are zero, and if there is a first minor distinct from zero corresponding to the elements of the second column, the results are precisely the same. But suppose that all the first minors corresponding to the elements of both the first and second columns are zero. Then suppose that a first minor corresponding to an element of the kth column is not zero. Then it follows from the form of (45) that when $a = a_j$, it will carry the factor μ^2; and let this minor be denoted by $\mu^2\Delta$. Then solving (44) we get

$$A_1 = \frac{\mu\Delta_1}{\mu^2\Delta} A_k,\ A_3 = \frac{\mu\Delta_2}{\mu^2\Delta} A_k,\ A_j = \frac{\mu^2\Delta_j}{\mu^2\Delta} A_k,\ j = 3, 4, \cdots \quad \infty.$$

where Δ_1, Δ_2, Δ_j, do not in general vanish when $\mu = 0$. It follows from the first two equations that A_k must carry μ as a factor, since the A_j are finite for $\mu = 0$. Hence in this case the y_{i1} have the same form as before. Similarly the y_{i2} have the same properties.

The solutions associated with $a_3^{(0)}$, $a_4^{(0)}$, $\cdots$, are found as in the preceding case. If there are several groups of $a_j^{(0)}$ in which these congruences exist the discussion must be made for each one separately.

VII

Solutions when $a_1^{(0)}$ is a multiple root.

Now suppose that two of the $a_j^{(0)}$ are equal and only two, and that there are none of the congruences treated in VI. Let us choose the notation so that $a_1^{(0)} = a_2^{(0)}$. From our work in the theory of linear differential equations

in infinitely many variables in constant coefficients and from IV, the solutions are of the form

$$(54) \qquad x_i = \sum_{j=1}^{\infty} A_j [c_{ij} e^{a_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k] \quad \text{or}$$

$$(55) \qquad x_i = A_1 [c_{i1} e^{a_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i1}^{(k)} \mu^k] + A_2 [(c_{i2} + t c_{i1}) e^{a_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k] + \sum_{j=1}^{\infty} A_j [c_{ij} e^{a_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k].$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the fundamental equation becomes either

$$(56) \qquad D(\sigma, \mu) = | \, [\sigma(c_{ij} e^{2a_j^{(0)}\pi} + \sum_{k=1}^{\infty} x_{ij}^{(k)} (2\pi) \mu^k) - c_{ij}(\sigma - 1)] \, | = 0 \quad \text{or}$$

$$(57) \qquad D(\sigma, \mu) = | \, [\sigma\{c_{i1} e^{2a_1^{(0)}\pi} + \sum_{k=1}^{\infty} x_{i1}^{(k)} (2\pi) \mu^k\} - c_{i1}(\sigma - 1)],$$
$$[\sigma\{(c_{i2} + 2\pi c_{i1}) e^{2a_1^{(0)}\pi} + \sum_{k=1}^{\infty} x_{i2}^{(k)} (2\pi) \mu^k\} - c_{i2}(\sigma - 1)], \cdots | = 0.$$

For $\mu = 0$ both of these, by the theory of infinite determinants, reduce to

$$(58) \qquad D_0 = | \, c_{ij} \, | \, [1 - \sigma(1 - e^{2a_1^{(0)}\pi})]^2 \prod_{j=3}^{\infty} [1 - \sigma(1 - e^{2a_j^{(0)}\pi})].$$

As before we set $a = a_1^{(0)} + \delta_1$, and expand as a power series in δ_1. Then we see that the term of lowest degree in δ_1 alone is of the second. When the determinant is of the form (56) with $a_1^{(0)} = a_2^{(0)}$, the term of lowest degree in μ alone is of the second in general. Then we have the same form as in VI. But if the determinant is of the form (57), the term in μ alone is in general of the first degree. In the former case we have a consideration similar to that in VI; in the latter case, in general the solutions for δ_1 are of the form

$$(59) \qquad \begin{aligned} \delta_{11} &= \mu^{1/2} P(\mu^{1/2}) \\ \delta_{12} &= \mu^{1/2} P(-\mu^{1/2}), \end{aligned}$$

where P is a power series in $\sqrt{\mu}$ and contains a term independent of μ. The discussion of the special cases is made just as in VI. On substituting these expansions for $a = a_1^{(0)} + \delta_1$ in (44) we solve for the A_j as power series in $\sqrt{\mu}$. These A_j substituted in (42) give y_{i1} and y_{i2} as power series in $\sqrt{\mu}$.

If p of the $a_j^{(0)}$ are equal, then for these roots the expansions of D starts with δ_1^p as the term of lowest degree in δ_1 alone, and except in the special cases corresponding to those mentioned in the foregoing, the term in μ alone is of the first degree. Consequently in general for $a_1^{(0)} = a_2^{(0)} \cdots = a_p^{(0)}$, we have

$$\delta_{ij} = \epsilon^j \mu^{1/p} P(\epsilon^j \mu^{1/p}), \; j = 0, 1, \cdots, p - 1,$$

where ϵ is a pth root of unity.

VIII

Solutions when there are equalities and congruences among the $a_j^{(0)}$.

Suppose that two of the $a_j^{(0)}$, for example $a_1^{(0)}$ and $a_2^{(0)}$ are equal, and that a third one, say $a_3^{(0)}$, differs from $a_1^{(0)}$ by an imaginary integer. Furthermore we shall assume that there are no other equalities or congruences among the $a_j^{(0)}$. Two cases arise here: (a) the solutions are of the form (54) with $a_1^{(0)} = a_2^{(0)}$; (b) the solutions are of the form (55)

Case (a). In this case we have

$$D_0 = |\, c_{ij} \,| \, [1 - \sigma(1 - e^{2a_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1 - \sigma(1 - e^{2aj^{(0)}\pi})].$$

In setting $a = a_1^{(0)} + \delta_1$, we find that the term in D of lowest degree in δ_1 alone is of the third. To get the term in D of lowest degree in μ alone we set $\delta_1 = 0$. Then it becomes evident at once that each of the first columns carry μ as a factor, while the remaining ones do not. Consequently the term of lowest degree in μ is of the third degree at least. Furthermore since the first three columns vanish when $\mu = \delta_1 = 0$ there are no terms of lower degree than the third in δ_1 and μ. Hence in general we see D of the form

$$D = \delta_1^3 + \gamma_{21}\delta_1^2\mu + \gamma_{03}\mu^3 + \cdots = 0. \tag{60}$$

The problem is now one of implicit functions. The details of the special cases must be treated as they arise. However, we make the general statement that since the roots of the cubic terms of (60) set equal to zero are in general distinct, it follows from the theory of implicit functions that the three values of δ_1 are in general expansible in integral powers of μ.

Case (b). In this case we have

$$D_0 = [1 - \sigma(1 - e^{2a_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1 - \sigma(1 - e^{2aj^{(0)}\pi}) = 0.$$

On introducing δ_1 as before, we find that the term of lowest degree in δ_1 alone is of the third degree. But when the terms involving μ are retained in D, only the first and third columns vanish when $\mu = \delta_1 = 0$, and consequently the expansion of D will contain a term in μ^2 alone. Furthermore, since the first and third columns vanish for $\mu = \delta_1 = 0$, there will be no terms of degree lower than the second in μ and δ_1. Hence in general D has the form

$$D = \delta_1^3 + \gamma_{11}\delta_1\mu + \gamma_{02}\mu^2 + \cdots = 0. \tag{61}$$

In the general case in which γ_{11} and γ_{02} are not zero, there is one solution in integral powers of μ and two in powers of $\sqrt{\mu}$.

When the roots $a_j^{(0)}$ have higher multiplicities and more congruences among them, we make a similar discussion.

Periodic Orbits on a Surface of Revolution.*

By Daniel Buchanan.

§ 1. *Introduction.* The object of this paper is to determine the periodic orbits described by a particle which moves, subject to gravity, on a smooth surface of revolution, the axis of which is vertical. Let us denote the equation of the surface by

$$x^2 + y^2 - 2pz + 2\epsilon\phi\ (x^2 + y^2, z) = 0, \tag{1}$$

where p is a positive constant, ϵ an arbitrary parameter, and ϕ a power series in $x^2 + y^2$ and z, converging for $x^2 + y^2$ and $|z|$ sufficiently small. There will be no loss of generality if we suppose that the constant term of ϕ is zero, otherwise it could be eliminated by a linear substitution for z, that is, by a translation of the xy-plane along the z-axis. Since the equation (1) vanishes with $x^2 + y^2$ and z, it may be solved, by the theory of implicit functions, for $x^2 + y^2$ as a power series in z converging for $|z|$ sufficiently small. Hence the equation (1) may be expressed as

$$F(x, y, z) = x^2 + y^2 - 2pz + 2\epsilon f(z) = 0, \tag{2}$$

where f is a power series in z converging for $|z|$ sufficiently small.

For $\epsilon = 0$ the equation (2) represents a vertical paraboloid of revolution. The generating parabola has its axis coinciding with the positive z-axis and its semi-latus-rectum equal to p. Periodic orbits are first constructed when the particle moves on this paraboloid. Then the analytic continuation of these orbits is made with respect to ϵ and in this way the orbits for the more general surface (2) are determined. It was for this reason that the parameter ϵ was introduced in (2).

The form of the equation of the surface (2) was suggested by a somewhat similar equation used by Poincaré in his memoir, *Sur les Lignes Géodésiques des Surfaces Convexes: Trans. Am. Math. Soc.*, vol. vi (July, 1905). Geodesic lines play the same rôle in Poincaré's memoir as periodic orbits do in this paper.

It will be readily seen that when $\epsilon = 0$ the problem here considered is somewhat analogous to the well-known problem of the spherical pendulum.†

* Presented to the American Mathematical Society, Sept. 5, 1918.

† For a complete discussion of the spherical pendulum, including the horizontal as well as the vertical motion, see Moulton's *Periodic Orbits*, chap. III, also *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXII (1911), pp. 338-365.

§ 2. *The Differential Equations.* If the particle is of unit mass and moves without friction, then the differential equations of motion are

$$x'' = X, \quad y'' = Y, \quad z'' = Z - g,$$

where X, Y, and Z are the normal reactions due to the surface, and g is the acceleration due to gravity. Since the surface is assumed to be smooth, the normal reactions at any point are proportional to the direction cosines of the normal at that point, and therefore the differential equations become

$$\left.\begin{aligned} x'' &= X = \lambda F_x = 2\lambda x, \\ y'' &= Y = \lambda F_y = 2\lambda y, \\ z'' &= Z - g = \lambda F_z - g = 2\lambda(-p + \epsilon f_z) - g, \end{aligned}\right\} \tag{3}$$

where λ is a factor of proportionality.

These equations admit the vis viva integral

$$x'^2 + y'^2 + z'^2 = 2g(c - z), \tag{4}$$

where c is the constant of integration.

Since the equation of constraint (2) does not contain t explicitly, the factor λ can be obtained by differentiating $F(x, y, z)$ twice with respect to t and eliminating x'', y'', and z'' from the result by means of the differential equations (3). Then on making use of the relations

$$x^2 + y^2 = 2pz - 2\epsilon f, \quad x'^2 + y'^2 = -z'^2 + 2g(c - z),$$

from (2) and (4), respectively, we find

$$2\lambda = \frac{2g(z - c) - g(p - \epsilon f_z) + z'^2(1 - \epsilon f_{zz})}{p^2 + 2pz - 2\epsilon(f + pf_z) + \epsilon^2 f_z^2}. \tag{5}$$

The part of 2λ which is independent of ϵ is

$$\lambda_0 = \frac{2g(z - c) - gp + z'^2}{p(p + 2z)}. \tag{6}$$

(A). Periodic Orbits on a Paraboloid of Revolution.

§3. *Proof of Existence of Periodic Orbits.* For $\epsilon = 0$ the surface (2) becomes

$$x^2 + y^2 - 2pz = 0, \tag{7}$$

and the differential equations (3) become

$$x'' = \lambda_0 x, \quad y'' = \lambda_0 y, \quad z'' = -p\lambda_0 - g, \tag{8}$$

where λ_0 is defined in (6). The last equation of (8) is independent of the first two and admits the integral

$$z'^2 = \frac{4gz(c-z) + c_1}{p + 2z}, \tag{9}$$

where c_1 is the constant of integration.

If the vertical motion is to be periodic, z cannot increase indefinitely and therefore z' must vanish for some value of t, $t = t_0$, say. Suppose $z = z_0$ at $t = t_0$. Then since $z' = 0$ at $t = t_0$, it follows that the constant of integration in (9) has the value

$$c_1 = -4gz_0(c - z_0).$$

Hence the integral (9) becomes

$$z' = \pm\sqrt{\frac{4g}{p + 2z}(z_0 - z)(z_0 - c + z)}. \tag{10}$$

It is readily seen from (10) that z' vanishes for $z = z_0$, and also for $z = c - z_0$.

As p is assumed to be positive, no part of the paraboloid (7) will lie below the xy-plane. Then z and z_0 cannot be negative and therefore, for real initial conditions, c cannot be less than z_0. Three cases arise according to the values assigned to c. They are:

Case I. $z_0 \leqq c < 2z_0$.
Case II. $c = 2z_0$.
Case III. $c > 2z_0$.

Case I. Suppose $z_0 \leqq c < 2z_0$ or $c - z_0 < z_0$. Then z' is real so long as z lies in the interval $z_0 \leqq z \leqq c - z_0$. If the particle is started in the plane $z = z_0$, it cannot remain in this plane for $z = z_0 \neq 0$ does not satisfy the z-equation of (8). Moreover, it cannot move above this plane since $z < z_0$, hence it must move below this plane. Consequently z decreases or z' is negative, and the negative sign must be taken in (10). The particle continues to fall until it reaches the plane $z = c - z_0$ where the velocity again vanishes. As $z = c - z_0$ is not a solution of the z-equation in (8) and as z cannot be less than $c - z_0$ it must increase, or the positive sign must be taken in (10). The particle then rises until the plane $z = z_0$ is reached where the velocity again vanishes and the radical in (10) changes sign. Hence the particle oscillates between the two horizontal planes $z = z_0$ and $z = c - z_0$, the latter plane being the lower plane.

Case II. If $c = 2z_0$ or $c - z_0 = z_0$, then z' would be real only when $z = z_0$ and would therefore be identically zero. In this case the particle would revolve in the horizontal plane $z = z_0$, the centrifugal force and the reaction of the surface being just sufficient to overcome gravity.

Case III. If $c > 2z_0$ or $c - z_0 > z_0$ it can be shown by an argument similar to that used in discussing Case I that the particle oscillates between the same two planes $z = z_0$ and $z = c - z_0$, as in Case I, but the latter plane is the higher plane.

Thus if c is positive and not less than z_0, the vertical motion of the particle is periodic, and therefore the last equation of (8) must admit a periodic solution.

It could also be shown by the usual analytic existence proof that the last equation of (8) admits a periodic solution. It is not necessary, however, to establish the existence of a periodic solution by either method for, by Macmillan's theorem, quoted below, it is shown that if the formal construction of a periodic solution can be made, then this solution will converge under suitable conditions. Macmillan's theorem is as follows:*

If

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu, t) \quad (i = 1, \cdots, n) \tag{11}$$

is a system of differential equations in which the f_i are expansible as power series in $x_1, \cdots, x_n$, and μ, vanishing for $x_1 = \cdots = x_n = \mu = 0$, with coefficients which are uniform, continuous, and periodic functions of t with the period $2j\pi$: and if the f_i converge for $0 \leqq t \leqq 2j\pi$, when $|x_j| < \rho_j$, $|\mu| < r$, then the solutions $x_i(t)$ are expansible as power series in μ, or any fractional power of μ, which converge for all t in the interval $0 \leqq t \leqq 2j\pi$ provided $|\mu|$ is sufficiently small. If the constants of integration can be determined at each step so as to make the solution formally periodic with the period $2j\pi$, then the solutions so determined will be periodic and converge for all finite values of t provided $|\mu|$ is sufficiently small.

§ 4. *The Vertical Motion in terms of Elliptic Integrals.* The integral (10) may be written

$$\int \frac{\sqrt{p + 2z}\, dz}{\sqrt{(z_0 - z)(z - c + z_0)}} = 2\sqrt{g} \int_{t_0}^{t} dt = 2\sqrt{g}(t - t_0). \tag{10'}$$

We desire to express this integral in terms of elliptic integrals. The neces-

* *Trans. Am. Math. Soc.*, Vol. XIII, No. 2, pp. 146-158.

sary substitution is found to be

$$p+2z=\frac{\sqrt{mn}(1-\kappa w)}{1+\kappa w},$$

where

$$m=p+2z_0,\ n=p+2c-2z_0,\ \kappa=\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}.$$

Since p, c, and z_0 are all positive, then m is also positive. If $z_0 < p/2+c$ then n is likewise positive, and therefore κ is real and less than 1 numerically. When the above transformation is made in (10′) we obtain

$$-\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\int\frac{(1-\kappa w)\ dw}{(1-\kappa w)\sqrt{(1-w^2)(1-\kappa^2w^2)}}=2\sqrt{g}(t-t_0),$$

which simplifies to

$$\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\left\{I_1-2I_3-\frac{2\kappa}{1-\kappa^2}\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right\}=2\sqrt{g}(t-t_0),$$

where

$$I_1=\int\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the first kind, and

$$I_3=\int\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the third kind. Thus t can be found for any given value of w or z. For most purposes, however, it is more convenient to have z expressed as a function of t. This will be discussed in the next section.

The half-period, $T/2$, of a complete oscillation can be obtained by integrating (10′) between the limits z_0 and $c-z_0$. The corresponding limits for w are -1 and $+1$. Hence

$$2\sqrt{g}(T/2)=\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\left\{\int_{-1}^{+1}\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}}-2\int_{-1}^{+1}\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}}-\frac{2\kappa}{1-\kappa^2}\left[\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right]_{-1}^{+1}\right\}.$$

Since $$\left[\sqrt{\frac{1-w^2}{1-\kappa^2 w^2}}\right]_{-1}^{+1} = 0, \qquad \text{then}$$

$$T = \frac{4\sqrt{mn}}{\sqrt{g}(\sqrt{m}+\sqrt{n})}\left\{\int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-\kappa^2 w^2)}} - 2\int_0^1 \frac{dw}{(1-\kappa^2 w^2)\sqrt{(1-w^2)(1-\kappa^2 w^2)}}\right\}.$$

§ 5. *Construction of a Periodic Solution for the Vertical Motion.* Let

$$z = v + c/2, \tag{12}$$

where v is the new dependent variable. Then the last equation of (8) becomes

$$v'' = -\frac{4gv + v'^2}{k + 2v},$$

where $k = p + c$. This equation can be expanded in the form

$$v'' = -(1/k)(4gv + v'^2)[1 - 2v/k + 4v^2/k^2 + \cdots (-2v/k)^j + \cdots], \tag{13}$$

which will converge for $|v| < |k/2|$, or $z < p/2 + c$. Hence the particle must not rise above the plane $z = p/2 + c$, being the plane at the distance c above the horizontal plane passing through the focus of the generating parabola of (7).

Now let

$$v = \gamma w, \; t - t_0 = \tfrac{1}{2}\sqrt{k/g\;(1+\delta)}\,\tau, \tag{14}$$

where γ and δ are arbitrary parameters. When (14) is substituted in (13), the factor γ can be cancelled off in the resulting equation, and we obtain

$$\ddot{w} = -[(1+\delta)w + \mu\dot{w}^2][1 - 2\mu w + \cdots(-2\mu w)^j + \cdots], \tag{15}$$

where $\mu = \gamma/k$, and the dots denote derivation with respect to τ.

The terms of (15) which are independent of μ and δ are $\ddot{w} + w = 0$, and the period of the solution is 2π in τ. Now an existence proof, based on Poincaré's extension to Cauchy's theorem, would show that a periodic solution of (15) exists having the form,

$$w = \sum_{j=0}^{\infty} w_j \mu^j, \quad \delta = \sum_{j=1}^{\infty} \delta_j \mu^j, \tag{16}$$

where the w_j are periodic functions of τ with the period 2π, and the δ_j are constants; and that this solution converges for all values of τ in the interval $0 \leqq \tau \leqq 2\pi$, provided $|\mu|$ is sufficiently small. Instead of making this rather

simple existence proof, we shall assume the form of solution (16), make the formal construction, and then show that Macmillan's theorem will apply to prove the convergence of the solution.

Since

$$z = v + c/2 = \gamma w + c/2 = k\mu w + c/2, \tag{17}$$

and since $z' = 0$ at $t = t_0$, then at $\tau = 0$ (or $t = t_0$) we may choose,

$$w(0) = a, \ \dot{w}(0) = 0,$$

where a is a real constant. When these initial conditions are imposed on (16), we have

$$w_0(0) = a, \ w_j(0) = 0, \ j = 1, \cdots \infty, \ \dot{w}_j(0) = 0, \ j = 0, 1, \cdots \infty. \tag{18}$$

Let (16) be substituted in (15) and let the resulting equation be denoted by (15′). This equation is an identity in μ and we may equate the coefficients of the same powers of μ, thus obtaining differential equations which define the various w_j. These equations are to be integrated, and the constants of integration and the various δ_j are to be chosen so that the solutions will be periodic and satisfy the initial conditions (18).

Step 0: *Coefficient of* μ^0. The terms of (15′) that are independent of μ give the equation $\ddot{w}_0 + w_0 = 0$, and its solution which satisfies (18) is $w_0 = a \cos \tau$.

Step 1: *Coefficient of* μ. The differential equation obtained from the terms in μ is

$$\ddot{w}_1 + w_1 = -\delta_1 w_0 + 2w_0^2 - \dot{w}_0^2 = -a\delta_1 \cos \tau + (a^2/2)(1 + 3 \cos 2\tau). \tag{19}$$

In order that w_1 shall be periodic the coefficient of $\cos \tau$ in the right member of (19) must be zero, otherwise the solution would contain the non-periodic or Poisson term $\tau \sin \tau$. On putting $\delta_1 = 0$, the general solution of (19) becomes

$$w_1 = A_1 \cos \tau + B_1 \sin \tau + (a^2/2)(1 - \cos 2\tau).$$

From the initial conditions (18) it follows that the constants of integration A_1 and B_1 are both zero, and the desired solution at this step becomes

$$\delta_1 = 0, \ w_1 = (a^2/2)(1 - \cos 2\tau).$$

Step 2: *Coefficient of* μ^2. The differential equation at this step is

$$\begin{aligned} \ddot{w}_2 + w_2 &= -\delta_2 w_0 + 4w_0 w_1 - 4w_0^3 + 2w_0 \dot{w}_0^2 - 2\dot{w}_0 \dot{w}_1 \\ &= -a(\delta_2 + \tfrac{1}{2}a^2) \cos \tau - \tfrac{7}{2}a^3 \cos 3\tau, \end{aligned}$$

and the desired solution is

$$\delta_2 = -\tfrac{1}{2}a^2, \quad w_2 = \tfrac{7}{16}a^3(-\cos\tau + \cos 3\tau).$$

The remaining steps of the integration are entirely similar to the preceding step and by an induction to the general term it can readily be shown that the process can be carried on indefinitely. It will be observed that δ_j is zero if j is odd, and that each w_j carries the factor a^{j+1} and is a sum of cosines of multiples of τ having the opposite parity of j. The highest multiple of τ in w_j is $(j+1)$.

So far as the computation has been made, the solution of (15) is

$$\left.\begin{aligned} w &= a\cos\tau + (\mu a^2/2)(1-\cos 2\tau) - \frac{7\mu^2 a^3}{16}(\cos\tau - \cos 3\tau) + \cdots, \\ \delta &= -\tfrac{1}{2}a^2\mu^2 + \cdots. \end{aligned}\right\} \quad (20)$$

The convergence of this solution will now be considered. In order to put the differential equation (15) in the same form as (11) of Macmillan's theorem, we let

$$w = W_1, \quad dW_1/d\tau = W_2,$$

then (15) becomes

$$\begin{aligned} dW_1/d\tau &= f_1(W_1, W_2; \mu, \tau) = W_2, \\ dW_2/d\tau &= f_2(W_1, W_2; \mu, \tau) = -[(1+\delta)W_1 + W_2^2][1 - 2\mu W_1 + \cdots]. \end{aligned}$$

The f_1 and f_2 are obviously expansible as power series in W_1, W_2, and μ, and vanish for $W_1 = W_2 = \mu = 0$. The coefficients are constants and therefore satisfy the condition that they shall be uniform, continuous, and periodic functions of τ. Further, f_1 and f_2 converge for W_1, W_2, and μ sufficiently small numerically. Hence equations (15) satisfy the conditions of Macmillan's theorem and consequently the solution (20) converges for $|\mu|$ sufficiently small.

When (20) is substituted for w in (17) we obtain

$$z = c/2 + k\left[\mu a\cos\tau + \frac{\mu^2 a^2}{2}(1-\cos 2\tau) - \frac{7\mu^3 a^3}{16}(\cos\tau - \cos 3\tau) + \cdots\right].$$

As μ and a are both arbitrary and as they occur only in products as indicated, we may suppress either without loss of generality. Let us suppose $a = 1$. Then the periodic solution for the vertical motion of the particle becomes

$$\left.\begin{aligned} z &= c/2 + k[\mu\cos\tau + (\mu^2/2)(1-\cos 2\tau) - (7\mu^3/16)(\cos\tau - \cos 3\tau) \\ &\qquad + \cdots], \\ \delta &= -\tfrac{1}{2}\mu^2 + (\ )\mu^4 + \cdots. \end{aligned}\right\} \quad (21)$$

In the expansion for z, the multiples of τ in the coefficient of μ^j have the same parity as j and therefore

$$z(\pi) = c/2 - k\mu = c/2 - \gamma.$$

Further

$$z(0) = c/2 + k\mu = c/2 + \gamma.$$

Hence the particle oscillates between the planes $z = c/2 + \gamma$ and $z = c/2 - \gamma$. The parameter γ is therefore a scale factor denoting the amplitude of the oscillation on either side of the plane $z = c/2$.

§6. *The Horizontal Motion.* When the solution (21) has been substituted in the first two equations of (8) and the transformation

$$t - t_0 = \tfrac{1}{2}\sqrt{(k/g)(1+\delta)}\,\tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

has been made, we obtain

$$\ddot{x} + k/p[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots]\,x = 0, \tag{22}$$

and the same equation in y, where

$$\theta_1 = -\cos\tau, \quad \theta_2 = -\tfrac{1}{8}(1 - \cos 2\tau),$$

and the remaining θ_j are likewise sums of cosines of multiples of τ having the same parity as j, the highest multiple being j.

Equation (22) is similar to the one first discussed by Hill* in 1877 in his celebrated memoir on the motion of the lunar perigee. A very complete list of references to the literature of the differential equations of this type is given by Baker on page 134 of his memoir "On Certain Linear Differential Equations of Astronomical Interest."†

Three cases arise in the solution of (22) depending upon the values of $(k/4p)$.‡ They are

Case I. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ not an integer.
Case II. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ an integer.
Case III. $\tfrac{1}{2}\sqrt{k/p} = 0$.

Since $k = p + c$ and since p and c are both positive in the physical problem under consideration, it follows that $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and Case III need not be considered.

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* *The Collected Works of G. W. Hill*, Vol. I, pp. 243-270; *Acta Mathematica*, Vol. VIII, pp. 1-36.

† *Phil. Trans. of the Royal Society of London*, Series A, Vol. 216, pp. 129-186.

‡ Compare also Moulton's *Periodic Orbits*, Chap. III, § 52.

Case I.—This may be regarded as the general case.

The form of the solution of (22), first given by Floquet,* is

$$x = e^{i\alpha\tau}u, \tag{23}$$

where α and u are power series in μ, the former having constant coefficients, the latter periodic coefficients with the period 2π in τ. When (23) is substituted in (22) we obtain

$$\ddot{u} + 2i\alpha\dot{u} - \alpha^2 u + k/p[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots]u = 0. \tag{24}$$

Now let

$$\begin{aligned} \alpha &= \tfrac{1}{2}\sqrt{k/p} + \alpha_1\mu + \alpha_2\mu^2 + \cdots, \\ u &= u_0 + u_1\mu + u_2\mu^2 + \cdots, \end{aligned} \tag{25}$$

be substituted in (24). We obtain a differential equation (24′), which is an identity in μ. Since the right number is zero, then the coefficient of each power of μ must also be zero. On solving the various differential equations thus obtained, we determine the u_j and α_j, the u_j as periodic functions of τ having the period 2π, and the α_j as constants so determined that the u_j shall be periodic.

Since the solution (23) is later multiplied by an arbitrary constant, see equations (39), we may choose $u(0) = 1$, from which it follows that

$$u_0(0) = 1,\ u_j(0) = 0,\ j = 1, \cdots \infty. \tag{26}$$

Step 0: *Coefficient* of μ^0. From the terms of (24′) that are independent of μ, we have the differential equation

$$\ddot{u}_0 + i\sqrt{k/p}\ \dot{u}_0 = 0,$$

and its solution is

$$u_0 = a_0 + b_0 e^{-i\sqrt{k/p}\ \tau}, \tag{27}$$

where a_0 and b_0 are the constants of integration. Since $\sqrt{k/p}$ is not an integer in this case, the term $e^{-i\sqrt{k/p}\tau}$ does not have the period 2π and we put $b_0 = 0$. From (26) it follows that $a_0 = 1$ and the desired solution at this step becomes $u_0 = 1$.

Step 1: *Coefficient of* μ. On equating to zero the terms in μ in (24′) we obtain

$$\ddot{u}_1 + i\sqrt{k/p}\ \dot{u}_1 = \sqrt{k/p}\ \alpha_1 + k/p\ \cos\tau. \tag{28}$$

* *Annales de l'École Normale Supérieure*, 1883-4.

The complementary function is

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau},$$

a_1 and b_1 being arbitrary constants.

It is well known in the theory of differential equations that the presence of any term in the right member which has *exactly* the same period as any term of the complementary function will yield Poisson terms in the particular integral, that is, terms containing τ outside of trigonometric or exponential symbols. Since the constant part of the right member of (28) has *exactly* the same period as the constant in the complementary function, then non-periodic terms will arise in the particular integral unless $a_1 = 0$. On putting $a_1 = 0$ the complete solution of (28) becomes

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau} + \frac{k}{k-p}[\cos\tau - i\sqrt{k/p}\sin\tau]. \tag{29}$$

From the periodicity and initial conditions it follows that

$$b_1 = 0,\ a_1 = \frac{k}{p-k}$$

and the desired solution at this step becomes

$$u_1 = \frac{k}{p-k}[1 - \cos\tau + i\sqrt{k/p}\sin\tau],\ a_1 = 0. \tag{30}$$

Step 2: *Coefficient of* μ^2. The differential equation at this step is

$$\ddot{u}_2 + i\sqrt{k/p}\,\dot{u}_2 = a_2\sqrt{k/p} + \frac{k}{8p} - \frac{k^2}{2p(p-k)} + \frac{k^2}{p(p-k)}\left(\cos\tau - \tfrac{1}{2}\cos 2\tau + \tfrac{1}{2}i\sqrt{k/p}\sin 2\tau\right) - (k/p)\cos 2\tau\ , \tag{31}$$

and the solution which satisfies the periodicity and initial conditions is

$$\left.\begin{aligned} u_2 &= \frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}(\cos\tau - i\sqrt{k/p}\sin\tau) \\ &+ \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau + i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau, \\ a_2 &= \tfrac{1}{8}\sqrt{k/p}\left(\frac{5k-p}{p-k}\right). \end{aligned}\right\} \tag{32}$$

So far as the computation has been carried out it is observed that the

a_j are real constants and that the u_j consist of sums of cosines and i times sines of multiples of τ, the highest multiple being j. It will now be shown by induction that these properties hold in general.

Let us suppose that $a_1, \cdots, a_{n-1}, u_0, u_1, \cdots, u_{n-1}$ have been computed and that the various a_j are real and that the u_j, $j = 1, \cdots, n-1$, have the form

$$u_j = \sum_{l=0}^{j} (A_j^{(l)} \cos l\tau + iB_j^{(l)} \sin l\tau), \tag{33}$$

where the $A_j^{(l)}$ and $B_j^{(l)}$ are real constants. We wish to show that a_n can be determined as a real constant and that u_n has the same form as (33) when $j = n$.

Step n: Coefficient of μ^n. The differential equation at this step is

$$\ddot{u}_n + i\sqrt{k/p}\,\dot{u}_n = a_n\sqrt{k/p} + U_n(a_1, \cdots, a_{n-1}; u_1, \cdots, u_{n-1}). \tag{34}$$

The only undetermined constant which enters (34) is a_n and it is written explicitly in so far as it occurs. The function U_n is linear in $u_1, \cdots, u_{n-1}$. The terms of U_n which arise from $2ia\ddot{u}$ in (24) have the form

$$2ia_l u_{n-l}, \quad l = 1, \cdots, n-1,$$

and are cosines and i times sines of multiples of τ, the highest multiple being $n-1$. The terms which arise from a^2u in (24) have the form

$$-(\ )a_l a_m u_{n-(l+m)}, \quad l, m = 0, \cdots, n-1, \ l+m \leqq n-1;$$

where ( ) denotes 1 if $l = m$ and 2 if $l \neq m$. These terms have the same form as the preceding terms. When $l = m = 0$, the term $-a_0^2 u_n$ will cancel off with the term $(k/4p)u_n$ arising from the last factor of (24) and therefore the highest multiple of τ in the sines and cosines is $n-1$. The last factor of (24) gives, in addition to the term $(k/4p)u_n$ just considered, terms of the type

$$(k/p)\,\theta_l u_{n-l}, \quad l = 1, \cdots, n-1.$$

These terms are also of the same form as (33) but the highest multiple of τ which they yield is n. Therefore

$$U_n = \sum_{l=0}^{n} [a_l^{(n)} \cos l\tau + ib_l^{(n)} \sin l\tau], \tag{35}$$

where $a_l^{(n)}$ and $b_l^{(n)}$ are real constants.

Let us now determine the periodic solution of (34). In order that the

solution shall be periodic, the right member of (34) must contain no constant terms. Hence

$$a_n = -\sqrt{p/k}\, a_0^{(n)},$$

a real constant. The complete solution of (34) then becomes

$$u_n = a_n + b_n e^{-i\sqrt{k/p}\,\tau} + \bar{U}_n,$$

where a_n and b_n are the constants of integration, and the particular integral $\bar{U}_n$ has the same form as (35), except that it has no constant term. From the periodicity and the initial conditions we have

$$b_n = 0,\ a_n = -\bar{U}_n(0),$$

and the desired solution for u_n is the same as when $j = n$. Hence the properties of a_j and u_j already stated hold in general.

A second solution of (22) could be constructed in an entirely similar way but it is not necessary to make the construction as this solution can be obtained directly from the former solution.

Let the solution of (24) already obtained be denoted by $u(\tau, +i)$. Then one solution of (22) is

$$x = e^{+i\alpha\tau}\, u(\tau, +i). \tag{36}$$

Since the differential equation (22) is independent of i, a change in the sign of i in (36) will still give a solution, viz.,

$$x = e^{-i\alpha\tau}\, u(\tau, -i). \tag{37}$$

Thus a second solution can be obtained by changing the sign of i in the first solution.

The determinant formed by the two solutions (36) and (37) together with their derivatives with respect to τ is a constant,* and its value can be computed most readily when $\tau = 0$. This determinant is

$$D = -2[i\alpha + \dot{u}(0, +i)] = -i\sqrt{k/p} + \text{terms in } \mu, \tag{38}$$

which is different from zero for $\mu = 0$ and therefore remains different from zero for $|\mu|$ sufficiently small. The two solutions (36) and (37) therefore constitute a fundamental set and the most general solution of (22), as well as of the similar equation in y, is

* Moulton's *Periodic Orbits*, § 18.

$$
\left.\begin{aligned}
x &= A_1 e^{i\alpha\tau} u^{(1)} + A_2 e^{-i\alpha\tau} u^{(2)}, \\
y &= B_1 e^{i\alpha\tau} u^{(1)} + B_2 e^{-i\alpha\tau} u^{(2)}, \\
\alpha &= \tfrac{1}{2}\sqrt{k/p}\,[1 + \frac{5k-p}{4(p-k)}\,\mu^2 + \cdots], \\
u^{(1)} = u(\tau, +i) &= 1 + \frac{k}{k-p}[1 - \cos\tau + i\sqrt{k/p}\sin\tau]\mu \\
&+ \Big[\frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}(\cos\tau - i\sqrt{k/p}\sin\tau) \\
&+ \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau \\
&+ i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau\Big]\mu^2 + \cdots, \\
u^{(2)}(+i) &= u^{(1)}(-i),
\end{aligned}\right\} \quad (39)
$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

By using Macmillan's theorem, quoted in § 3, it can readily be shown that the above solutions converge for all τ in the interval $0 \leqq \tau \leqq 2\pi$ provided $|\mu|$ is sufficiently small.

Case II. Suppose $\sqrt{k/p} = \nu$, an integer. Since $k = p + c$ and p and c are both positive, then $\nu > 1$. Two sub-cases arise depending upon the value of ν. They are:

Sub-case I. $\nu = 2$,
Sub-case II. $\nu \neq 2$.

Sub-case I. The construction proceeds as in Case I until equation (27) is reached, and this becomes

$$u_0 = a_0 + b_0 e^{-2i\tau}, \quad (40)$$

where both terms have the period 2π. From the initial condition $u_0(0) = 1$ it follows that $b_0 = 1 - a_0$, and the solution for u_0 becomes

$$u_0 = a_0 + (1 - a_0)e^{-2i\tau}, \quad (41)$$

with the constant a_0 remaining arbitrary at this step.

Step 1: *Coefficient of* μ. When we equate to zero the coefficient of μ in (24″), that is in equation (24) after (25) has been substituted and $\sqrt{k/p}$ replaced by 2, we obtain

$$\ddot{u}_1 + 2i\dot{u}_1 = 2a_1a_0 - 2a_1(1 - a_0)e^{-2i\tau} + 2a_0e^{i\tau} + 2e^{-i\tau} + 2(1 - a_0)e^{-3i\tau}. \quad (42)$$

The complementary function of this equation is

$$u_1 = a_1 + b_1 e^{-2i\tau}.$$

Since terms of *exactly* the same period as those of the complementary function occur in the right member of (42), viz., constants and terms in $e^{-2i\tau}$, the particular integral will contain non-periodic terms unless the constants and the coefficient of $e^{-2i\tau}$ are put equal to zero. Since we seek a periodic solution we therefore put

$$2a_1 a_0 = 0, \quad 2a_1(1 - a_0) = 0. \tag{43}$$

These equations are satisfied by $a_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 or this case would be the same as Case I. The complete solution of (42) then becomes

$$u_1 = a_1 + b_1 e^{-2i\tau} - \tfrac{2}{3}a_0 e^{i\tau} + 2e^{-i\tau} - \tfrac{2}{3}(1 - a_0)e^{-3i\tau}. \tag{44}$$

Since $u_1(0) = 0$, then

$$b_1 = -(\tfrac{4}{3} + a_0).$$

At this step a_0 and a_1 still remain undetermined.

Step 2: Coefficient of μ^2. The terms of (24″) which contain the factor μ^2 give the equation

$$\begin{aligned}\ddot{u}_2 + 2i\dot{u}_2 = {} & [2a_2 a_0 + \tfrac{7}{6}a_0 + 2] + [-2a_2(1 - a_0) - \tfrac{7}{6}a_0 + 1\tfrac{9}{6}]e^{-2i\tau} \\ & + 2a_1 e^{i\tau} - 1\tfrac{0}{3}a_0 e^{2i\tau} - \tfrac{8}{3}e^{-i\tau} \\ & - 2(a_1 + \tfrac{4}{3})e^{-3i\tau} - 1\tfrac{0}{3}(1 - a_0)e^{-4i\tau}.\end{aligned}$$

On equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member, as at the preceding step, we have

$$\left.\begin{aligned} 2a_2 a_0 - \tfrac{7}{6}a_0 + 2 &= 0, \\ -2a_2(1 - a_0) - \tfrac{7}{6}a_0 + 1\tfrac{9}{6} &= 0. \end{aligned}\right\} \tag{45}$$

These equations are satisfied by

$$a_2 = \pm \tfrac{1}{12}\sqrt{217}, \quad a_0 = \tfrac{1}{2} \mp \tfrac{1}{14}\sqrt{217}, \tag{46}$$

where the upper signs are to be taken together, also the lower. When equations (46) are satisfied, the general solution for u_2 will be periodic, being

$$\begin{aligned} u_2 = {} & a_2 + b_2 e^{-2i\tau} - \tfrac{2}{3}a_1 e^{i\tau} + \tfrac{5}{12}a_0 e^{2i\tau} - \tfrac{8}{3}a_0 e^{-i\tau} \\ & + \tfrac{2}{3}(a_1 + \tfrac{4}{3})e^{-3i\tau} + \tfrac{5}{12}(1 - a_0)e^{-4i\tau}. \end{aligned}$$

Since $u_2(0) = 0$ then

$$b_2 = -a_2 + \tfrac{8}{3}a_0 - \tfrac{47}{36}.$$

The undetermined constants at this step are a_2 and a_1.

Step 3: *Coefficient of* μ^3. The differential equation at this step is

$$\ddot{u}_3 + 2iu_3 = U_3 = -2ia_3u_0 - 2ia_2u_1 + 2a_3u_0 + 2a_2u_1 \\ - u_0\theta_3 + u_1(\tfrac{1}{2} - 4\cos 2\tau) + 4u_2\cos\tau.$$

If we consider only the part of U_3 which involves the arbitrary constants that are determined at this step we have

$$\begin{aligned} U_3 = {} & 2a_3a_0 + a_1(2a_2 + \tfrac{7}{6}) + d_1 \\ & + [-2(1-a_0)a_3 + a_1(2a_2 - \tfrac{7}{6}) + d_2e^{-2i\tau}] \\ & + \text{terms in } e^{\pm i\tau},\ e^{2i\tau},\ e^{\pm 3i\tau},\ e^{\pm 5i\tau},\ e^{-4i\tau}, \end{aligned} \tag{47}$$

where d_1 and d_2 are known constants. To make u_3 periodic we equate to zero the constants and the coefficient of $e^{-2i\tau}$ in (47) giving

$$\left.\begin{aligned} 2a_3a_0 + a_1(2a_2 + \tfrac{7}{6}) + d_1 &= 0, \\ -2(1-a_0)a_3 + a_1(2a_2 - \tfrac{7}{6}) + d_2 &= 0. \end{aligned}\right\} \tag{48}$$

The determinant of the coefficients of a_3 and a_1 in (48) is

$$D_1 = \pm\tfrac{2}{3}\sqrt{217}, \tag{49}$$

where the upper sign here is to be taken with the upper signs in (46), similarly for the lower signs. Since this determinant D_1 is different from zero, equations (48) can be solved for a_3 and a_1, and the solutions are unique for either set of values in (46). When (48) is satisfied, the solution for u_3 will therefore be periodic. Two constants of integration, a_3 and b_3, say, will arise from the complementary function. One constant, b_3, say, will be determined from the initial condition $u_3(0) = 0$, while a_2 and a_3 remain undetermined at this step.

The succeeding steps of the integration are entirely similar to step 3 just considered. So far as the constants of integration are concerned one of them, b_j say, is determined by the initial condition $u_j(0) = 0$ at the step where it arises. The other constant a_j of the step j, is not determined until the step $j + 2$ is reached, where, on equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member of the differential equation in u_{j+2}, two linear equations in a_{j+2} and a_j are obtained and the determinant of their coefficients is the same as D_1 in (49). Hence a_{j+2} and a_j can be uniquely determined provided one set of values is taken in (46).

It is thus evident that two solutions of (24″) can be obtained in the foregoing construction, according as the upper or lower signs are taken in (46). The corresponding solutions of (22) are

$$x = x_1 = e^{i(1+\frac{1}{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(\tfrac{1}{2}-\tfrac{1}{14}\sqrt{217}) + (\tfrac{1}{2}+\tfrac{1}{14}\sqrt{217})e^{-2i\tau}+(\ )\mu+\cdots],$$

and

$$x = x_2 = e^{i(1-\frac{1}{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(\tfrac{1}{2}+\tfrac{1}{14}\sqrt{217}) + (\tfrac{1}{2}-\tfrac{1}{14}\sqrt{217})e^{-2i\tau}+(\ )\mu+\cdots].$$

If we proceed as in Case I to obtain two solutions by changing the sign of i in the preceding solutions it would appear that two additional solutions could be obtained, but this is impossible since the differential equation (22) is only of the second order and admits of but two solutions. This apparent difficulty is overcome if the factor $e^{i\tau}$ is multiplied into the part of the solutions contained in the square brackets [ ]. Thus

$$\left.\begin{aligned} x_1 &= e^{i(\frac{1}{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(\tfrac{1}{2}-\tfrac{1}{14}\sqrt{217})e^{i\tau} + \\ &\quad + (\tfrac{1}{2}+\tfrac{1}{14}\sqrt{217})e^{-i\tau}+(\ )\mu+\cdots], \\ x_2 &= e^{-i(\frac{1}{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(\tfrac{1}{2}+\tfrac{1}{14}\sqrt{217})e^{i\tau} + \\ &\quad + (\tfrac{1}{2}-\tfrac{1}{14}\sqrt{217})e^{-i\tau}+(\ )\mu+\cdots]. \end{aligned}\right\} \tag{50}$$

It is obvious, therefore, that the two solutions x_1 and x_2 differ only in the sign of i or of $\sqrt{217}$.

If the exponentials in the square brackets of (50) are expressed in trigonometric form, these solutions become

$$\left.\begin{aligned} &x_1 = e^{ia\tau}u^{(1)},\ x_2 = e^{-ia\tau}u^{(2)}, \\ &a = \tfrac{1}{12}\sqrt{217}\,\mu^2+\cdots, \\ &u^{(1)} = \cos\tau - i\,\tfrac{1}{7}\sqrt{217}\sin\tau+(\ )\mu+\cdots, \\ &u^{(2)}(+i) = u^{(1)}(-i). \end{aligned}\right\} \tag{51}$$

The terms represented by $u^{(1)}$ and $u^{(2)}$ differ not only in i but also in the sign of $\sqrt{217}$. They are power series in μ with sums of cosines and $i\sqrt{217}$ times sines of multiples of τ, the highest multiple in the coefficient of μ^j being $j+1$.

The determinant of the two solutions in (51) together with their derivatives is a constant, as in Case I, and its value at $\tau = 0$ is

$$D_2 = -2[ia + u^{(1)}(0)] = i\,\tfrac{2}{7}\sqrt{217}+(\ )\mu+\cdots,$$

which is different from zero for $\mu = 0$, and therefore remains different from zero for $|\mu|$ sufficiently small. Hence the two solutions (51) constitute a fundamental set, and the most general solutions of (22) and the similar equation in y are

$$x = A_1 e^{i\alpha\tau} u^{(1)} + A_2 e^{-i\alpha\tau} u^{(2)}, \quad y = B_1 e^{i\alpha\tau} u^{(1)} + B_2 e^{-i\alpha\tau} u^{(2)}, \tag{52}$$

where α, $u^{(1)}$, and $u^{(2)}$ are defined in (51), and A_1, A_2, B_1, B_2 are the constants of integration.

Sub-case II. When ν is an integer different from 2, the construction is the same as in the preceding sub-case until equations (43) are reached. The equations analogous to (43) are

$$\nu a_1 a_0 = 0, \qquad \nu a_1 (1 - a_0) = 0. \tag{53}$$

These are satisfied by $a_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 as in the preceding sub-case.

The equations which correspond to (45) are

$$\left.\begin{aligned} a_0 \left\{ \nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0, \\ (1 - a_0) \left\{ -\nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0. \end{aligned}\right\} \tag{54}$$

Since $a_0 \neq 1$ in Case II, then the solutions of (54) are

$$a_0 = 0, \; a_2 = \frac{\nu(5\nu^2 - 1)}{8(\nu^2 - 1)}. \tag{55}$$

The equations analogous to (48) are

$$a_1 \left\{ \nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} = 0, \qquad -\nu a_3 + \nu^2 \theta_3^{(0)} = 0, \tag{56}$$

where $\theta_3^{(0)}$ is the constant part of θ_3 in equation (22). The solutions of these equations are

$$a_1 = 0, \; a_3 = \nu \theta_3^{(0)}.$$

The constants a_2 and a_4 are determined in the same way at the next step as a_1 and a_3 were found in the preceding step. They have the same coefficient in the equations analogous to (56) as a_1 and a_3 have in (56), and similarly for the succeeding steps. It will be found that

$$a_j = 0, \; j = 0, 1, \cdots, \nu - 3,$$

and that the remaining a_j are, in general, different from zero.

So far as the computation has been made we find

$$\left.\begin{aligned} u_0 &= e^{-i\nu\tau}, \\ u_1 &= \frac{\nu^2}{\nu^2-1}\,e^{-i\nu\tau} + \nu^2/2\left[\frac{e^{-i(\nu-1)\tau}}{\nu-1} - \frac{e^{-i(\nu+1)\tau}}{\nu+1}\right], \\ u_2 &= \left[\frac{\nu^4}{(\nu^2-1)^2} + \frac{\nu^2}{8(\nu^2-4)}\right]e^{-i\nu\tau} - \frac{\nu^4}{2(\nu^2-1)}\left[\frac{e^{-i(\nu-1)\tau}}{\nu-1}\right. \\ &\quad \left. - \frac{e^{-i(\nu+1)\tau}}{\nu+1}\right] - \nu^2/32\left[\frac{e^{-i(\nu-2)\tau}}{\nu-2} - \frac{e^{-i(\nu+2)\tau}}{\nu+2}\right], \\ a_1 &= 0,\ a_2 = \frac{\nu(5\nu^2-1)}{8(\nu^2-1}. \end{aligned}\right\}$$

When these terms are substituted in (25) and the result in (23) we obtain one solution of (22). Let it be denoted by

$$x = e^{i\alpha\tau}u^{(1)}. \tag{58}$$

Another solution can be obtained by changing the sign of i in (58), thus

$$x = e^{-i\alpha\tau}u^{(2)}, \tag{59}$$

where $$u^{(2)}(+i) = u^{(1)}(-i).$$

The determinant of these two solutions and their first derivatives is

$$-2[i\alpha + iu^{(1)}(0)] = i\nu + (\ )\mu + \cdots,$$

which is different from zero for $|\mu|$ sufficiently small. The solutions (58) and (59) therefore constitute a fundamental set, and the general solutions of (22) and the corresponding equation in y are of the same form as (39) or (52), viz.,

$$x = A_1e^{i\alpha\tau}u^{(1)} + A_2e^{-i\alpha\tau}u^{(2)},\ y = B_1e^{i\alpha\tau}u^{(1)} + B_2e^{-i\alpha\tau}u^{(2)}, \tag{60}$$

where α and $u^{(1)}$ have the values in (58). If the exponentials in (58) are expressed in trigonometric form then

$$u^{(1)} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}[A^{(j)} + B^{(j)}_{\pm l}\cos(\nu \pm l)\tau + iC^{(j)}_{\pm l}\sin(\nu \pm l)\tau]\mu^j, \tag{61}$$

where $A^{(j)}$, $B^{(j)}_{\pm l}$, and $C^{(j)}_{\pm l}$ are real constants. In particular

$$A^{(j)} = 0,\ j = 0, 1, 2,\ B^{(j)}_{\pm l} = -C^{(j)}_{\pm l},\ j = 0, 1, 2,\ l = 0, 1, 2,$$

$$B^{(0)}_{0} = 1,\ B^{(1)}_{0} = -\frac{\nu^2}{\nu^2-1},$$

$$B^{(1)}_{-1} = \frac{\nu^2}{2(\nu-1)},\ B^{(1)}_{-1} = -\frac{\nu^2}{2(\nu+1)},$$

$$B_0^{(2)} = \frac{\nu^4}{(\nu^2-1)^2} + \frac{\nu^4}{8(\nu^2-4)},\quad B_{-1}^{(2)} = -\frac{\nu^4}{2(\nu-1)(\nu^2-1)},$$

$$B_{+1}^{(2)} = -\frac{\nu^4}{2(\nu+1)(\nu^2-1)},\quad B_{-2}^{(2)} = -\frac{\nu^2}{32(\nu-2)},$$

$$B_{+2}^{(2)} = \frac{\nu^2}{32(\nu+2)}.$$

From (57) it follows that

$$A^{(j)} = 0,\ j = 0, 1, \cdots, \nu-3.$$

The remaining $A^{(j)}$ are different from zero, in general.

If the factors $e^{i(\nu/2)\tau}$ and $e^{-i(\nu/2)\tau}$ are taken with $u^{(1)}$ and $u^{(2)}$, respectively, in (60), then $u^{(1)}$ and $u^{(2)}$ will have cosines and sines of $(\nu/2 \pm l)\tau$ instead of $(\nu \pm l)\tau$ as in (61).

§ 7. *The Arbitrary Constants.* Each of the solutions for x, y, and z contains two arbitrary constants, and the values of these constants for the physical problem will now be considered.

In this discussion no discrimination will be made between the two cases of the preceding section. It was for this reason that the same notation was chosen for the solutions (39), (52), and (60).

Besides satisfying the differential equations (8), the solutions for x, y, and z must also satisfy (7), viz.,

$$x^2 + y^2 - 2pz = 0, \tag{7}$$

and their derivatives with respect to t must satisfy the *vis viva* integral (4). When the transformation

$$t - t_0 = \tfrac{1}{2}\sqrt{k/g(1+\delta)}\,\tau,\ \delta = -\tfrac{1}{2}\mu^2 + \cdots$$

is made in (4), this integral becomes

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (k/2)(1+\delta)(c-z). \tag{62}$$

Since any horizontal section of the surface (7) is a circle with centre on the z-axis, the xy-axes may be rotated about the z-axis without changing the form of (7), and we may therefore suppose that the particle lies in the xz-plane at the initial time, or $y = 0$ at $\tau = 0$. Since $u^{(1)}(0) = u^{(2)}(0) = 1$, then it follows that $B_1 + B_2 = 0$.

As $\dot{z}(0) = 0$, the particle must be initially projected in a horizontal plane. Since any horizontal section of the surface of constraint is a circle with centre on the z-axis and since $u(0) = 0$, then it follows that $\dot{x}(0) = 0$. When this condition is imposed on the solution for x, we have

$$[ia + \dot{u}^{(1)}(0)]\ [A_1 - A_2] = 0.$$

The factor $[ia + \dot{u}^{(1)}(0)]$ is different from zero for $|\mu|$ sufficiently small and therefore

$$A_1 = A_2.$$

Hence the solutions for x and y become

$$x = A[e^{ia\tau}u^{(1)} + e^{-ia\tau}u^{(2)}],\ y = B[e^{ia\tau}u^{(1)} - e^{-ia\tau}u^{(2)}], \qquad (63)$$

where the now unnecessary subscripts on the constants A and B have been dropped.

On putting $\tau = 0$ in (63) and (21) we get

$$\left.\begin{aligned} &x(0) = 2A, && \dot{x}(0) = 0, \\ &y(0) = 0, && \dot{y}(0) = 2B[ia + \dot{u}^{(1)}(0)], \\ &z(0) = c/2 + k\mu, && \dot{z}(0) = 0. \end{aligned}\right\} \qquad (64)$$

When these values are substituted in (7) and (62) we find

$$A = \pm \tfrac{1}{2}\sqrt{p(c + 2k\mu)} = \pm \tfrac{1}{2}\sqrt{p(c + 2\gamma)},$$

$$B = \pm \frac{\sqrt{k(1+\delta)(c-2\gamma)}}{4[ia + \dot{u}^{(1)}(0)]} = \mp i\frac{\sqrt{k(1+\delta)(c-2\gamma)}}{4[a - i\dot{u}^{(1)}(0)]}.$$

Thus A is real and B is purely imaginary. Hence the solutions of equations (8), in terms of τ, are

$$\left.\begin{aligned} x &= \pm \tfrac{1}{2}\sqrt{p(c + 2\gamma)}\ [e^{ia\tau}u^{(1)} + e^{-ia\tau}u^{(2)}], \\ y &= \mp \frac{i\sqrt{k(1+\delta)(c-2\gamma)}}{4[a - i\dot{u}^{(1)}(0)]}\ [e^{ia\tau}u^{(1)} - e^{-ia\tau}u^{(2)}], \\ z &= c/2 + k[\mu\cos\tau + \tfrac{1}{2}\mu^2(1 - \cos 2\tau) \\ &\qquad - \tfrac{7}{16}\mu^3(\cos\tau - \cos 3\tau) + \cdots], \\ \delta &= -\tfrac{1}{2}\mu^2 + \cdots, \\ \mu &= \gamma/k, \qquad k = p + c. \end{aligned}\right\} \qquad (65)$$

The terms a, $u^{(1)}$, and $u^{(2)}$ are defined in (39), (52), and (60) according to the value of $\sqrt{k/p}$. The double signs in (65) depend upon the octants of space into which the particle is initially projected.

If the exponentials in (65) are expressed in trigonometric form, the solutions of (22) for x and y become

$$x = \sqrt{p(c + 2\gamma)}\ [\{\ \}_1\cos a\tau - \{\ \}_2\sin a\tau],$$

$$y = \frac{\sqrt{k(1+\delta)(c-2\gamma)}}{2[a - i\dot{u}^{(1)}(0)]}\ [\{\ \}_1\sin a\tau + \{\ \}_2\cos a\tau],$$

where $\{\ \}_1$ and $\{\ \}_2$ are power series in μ which have different values for the different cases of § 6. For Case I,

$$\{\ \}_1 = 1 + \sum_{j=1}^{\infty} \sum_{l=0}^{j} [a_{jl} \cos l\tau]\mu^j,$$

$$\{\ \}_2 = \sum_{j=1}^{\infty} \sum_{l=1}^{j} [b_{jl} \sin l\tau]\mu^j,$$

where a_{jl} and b_{jl} are the coefficients of $\cos l\tau$ and $\sin l\tau$, respectively, in u_j, equations (30), (32), and (33).

For Case II,

$$\{\ \}_1 = \sum_{j=0}^{\infty} \sum_{l=0}^{j} [A^{(j)} + B^{(j)}_{\pm l} \cos(\nu \pm l)\tau]\mu^j,$$

$$\{\ \}_2 = \sum_{j=1}^{\infty} \sum_{l=0}^{j} [C^{(j)}_{\pm l} \sin(\nu \pm l)\tau]\mu^j.$$

The constants $A^{(j)}$, $B^{(j)}_{\pm l}$, and $C^{(j)}_{\pm l}$ have the same values as in (61) for sub-case II, and for sub-case I they are, in so far as the computation has been carried out,

$$A^{(0)} = 0,\ B^{(0)}_0 = 1,\ C^{(0)}_0 = -\tfrac{1}{7}\sqrt{217}.$$

There still remain in the solutions (65) two constants which have not been determined. They are c and γ $(= k\mu)$. Since c is the constant of integration arising in the *vis viva* integral, its value depends upon the initial velocity of the particle. As $x' = z' = 0$ at $t = t_0$, the initial velocity may be denoted by y'_0. Then from the *vis viva* integral (4) it follows that

$$y'^2_0 = 2g(c - z_0) = g(c - 2\gamma),\ \text{or}\ c - 2\gamma = y'^2_0/g.$$

Since, further, $c + 2\gamma = 2z_0$, then

$$c = z_0 + y'^2_0/2g,\ \ \gamma = \tfrac{1}{2}(z_0 - y'^2_0/2g).$$

Thus the constants c and γ are functions of the initial velocity and the initial height.

Let us now return to the three cases of § 3. Since $z_0 = c/2 + \gamma$ these cases become:

Case I. $\quad 0 < \gamma \leqq c/2,$

Case II. $\quad \gamma = 0,$

Case III. $\quad \gamma < 0,\ |\gamma| \leqq c/2.$

In Case I and Case III the particle oscillates between the horizontal planes $z = c/2 + \gamma$ and $z = c/2 - \gamma$. The two orbits are geometrically the same but the motion in the one orbit is half a period ahead of the motion in the

other. If $\gamma = c/2$, then the initial velocity is zero and as there is no lateral projection the particle will move in the vertical parabola $x^2 = 2pz$. If $\gamma = -c/2$, then $z_0 = 0$, and the particle is projected from the lowest point with the initial velocity $y'_0 = \sqrt{2gc}$. It will therefore move in the vertical parabola $y^2 = 2pz$ which is dynamically the same orbit as $x^2 = 2pz$. The orbits when $\gamma = \pm c/2$ correspond therefore to the simple pendulum.

In Case II when $\gamma = 0$, then $z = c/2$, a constant, and the differential equations of motion become

$$x'' + (g/p)x = 0, \quad y'' + (g/p)y = 0, \quad z'' = 0.$$

Their periodic solutions which satisfy the initial conditions are

$$x = \sqrt{pc}\,\cos\sqrt{g/p}\,t, \quad y = \sqrt{pc}\,\sin\sqrt{g/p}\,t, \quad z = c/2.$$

In this case the particle moves in a circle the plane of which is parallel to the xy-plane and at a distance $c/2$ above it.

In the first paragraph of § 5 it is stated that the inequality $z < p/2 + c$ must hold in order that the expression in (13) will converge. Now the maximum value of z is

$$z = c/2 + |\gamma|,$$

and therefore the above inequality becomes

$$c/2 + |\gamma| < p/2 + c,$$

or

$$|\gamma| < \tfrac{1}{2}(p + c).$$

Since $|\gamma|$ must not exceed $c/2$ in order that the initial velocity shall be real, it follows that the inequality $z < p/2 + c$ will always be satisfied for real initial conditions.

(B). Periodic Orbits on a Surface of Revolution.

§ 8. *The Method of Solution.*—Let us next consider the construction of the periodic orbits described on the more general surface of revolution represented by the equation

$$F(x, y, z) = x^2 + y^2 - 2pz + 2\epsilon f(z) = 0. \tag{2}$$

The differential equations of motion have already been found, equations (3) and (5). The method of constructing the periodic solution of these equations is to first make the analytic continuation with respect to ϵ of the periodic solution for the vertical motion obtained in (*A*) where $\epsilon = 0$, and then substitute this solution for z in the first two equations in (3). We thus

obtain two differential equations having periodic coefficients somewhat similar to (22).

§ 9. *The Equation of Variation.*—Let us substitute in the last equation of (3)

$$\left.\begin{aligned} z = \bar{z} + \xi,\ t - t_0 = \tfrac{1}{2}\sqrt{k/g(1+\delta)}\,\tau, \\ \delta = -\tfrac{1}{2}\mu^2 + \cdots, \end{aligned}\right\} \tag{66}$$

where $\bar{z}$ denotes the solution obtained in (21), and ξ is a function of τ which vanishes with ϵ. We obtain

$$\ddot{\xi} + \Theta_1\dot{\xi} + \Theta_2\xi = Z_0 + \epsilon Z_1 + \cdots + \epsilon^j Z_j + \cdots, \tag{67}$$

where the undefined terms have the following properties:

(1). The functions Θ_1 and Θ_2 are periodic functions of τ, but we shall show that it is not necessary to know the explicit values of these functions in order to solve (67).

(2). The functions $Z_0, Z_1, \cdots, Z_j$ denote power series in ξ having coefficients which are power series in μ with sums of cosines of multiples of τ in their coefficients. These functions also contain additional terms in $\dot{\xi}$ and $\dot{\xi}^2$, the former being multiplied by power series in μ with sums of sines of multiples of τ in the coefficients, the latter by similar series except that they contain cosines. The function Z_0 contains no linear terms in ξ or $\dot{\xi}$.

If we neglect the right member of (67) we obtain

$$\ddot{\xi} + \Theta_1\dot{\xi} + \Theta_2\xi = 0, \tag{68}$$

which is called *the equation of variation.* The generating solution is $z = \bar{z}$, or the expression for z in (21).

Now it has been shown by Poincaré * that if the generating solution contains an arbitrary constant which does not occur in the original differential equations of motion, viz., equations (3), then a solution of the equations of variation can be obtained by differentiating the generating solution with respect to this constant. Three constants occur in (21), viz., c, μ, and t_0, the latter entering implicitly through τ. The constant c occurs in 2λ, equation (5), and therefore enters the differential equations (3). The two remaining constants, μ and t_0, do not occur in (3) and therefore may be used in applying Poincaré's theorem. Each constant yields a solution of (68) and since the differential equation is only of the second order, both its solutions may be obtained from the generating solution. Hence it is not necessary to know the coefficients Θ_1 and Θ_2 in (68) in order to solve the differential equation.

* *Les Méthodes Nouvelles de la Mécanique Celeste*, Vol. I, chap. IV. *Loc. cit.*

Consider, first, the constant t_0. Then one solution of (68) is

$$\xi = \partial\bar{z}/\partial t_0 = -2\sqrt{\frac{g}{k(1+\delta)}}\,\partial\bar{z}/\partial t = 2\sqrt{\frac{g}{k(1+\delta)}}\,\mu S(\tau), \tag{69}$$

where

$$S(\tau) = \sin\tau - \mu\sin 2\tau - 7/16\mu^2\,(\sin\tau - 3\sin 3\tau) + \cdots. \tag{70}$$

Since this solution is later multiplied by an arbitrary constant, see equation (76), we may drop the constant factor of $S(\tau)$ in (69) and take

$$\xi = S(\tau) \tag{71}$$

as the solution.

Considering the constant μ, we obtain as the second solution of (68)

$$\xi = \frac{\partial\bar{z}}{\partial\mu} = \left(\frac{\partial\bar{z}}{\partial\mu}\right) + \frac{\partial\bar{z}}{\partial\tau}\frac{\partial\tau}{\partial\delta}\frac{\partial\delta}{\partial\mu}, \tag{72}$$

where the parentheses ( ) denote differentiation in so far as μ occurs explicitly in $\bar{z}$. Now

$$\left.\begin{aligned} \frac{\partial\bar{z}}{\partial\mu} &= kC(\tau) = k[\cos\tau + \mu(1-\cos 2\tau) \\ &\qquad - \frac{21\mu^2}{16}(\cos\tau - \cos 3\tau) + \cdots], \\ \frac{\partial\tau}{\partial\delta} &= -\tfrac{1}{2}\frac{\tau}{1+\delta},\quad \frac{\partial\mu}{\partial\delta} = \mu[-1 + \text{power series in } \mu^2]. \end{aligned}\right\} \tag{73}$$

Therefore the solution (72) becomes

$$\xi = k[C(\tau) + \tau KS(\tau)],$$

where

$$K = \mu^2[-1 + \text{power series in } \mu^2].$$

As in the previous solution we may drop the constant factor k and take

$$\xi = C(\tau) + \tau KS(\tau), \tag{74}$$

as the second solution of (68).

The two solutions (71) and (74) constitute a fundamental set, since the determinant of these two solutions together with their derivatives is different from zero for $|\mu|$ sufficiently small, being

$$\Delta = -1 + \text{power series in } \mu. \tag{75}$$

Hence the general solution of (68) is

$$\xi = N_1 S(\tau) + N_2[C(\tau) + \tau KS(\tau)], \tag{76}$$

where N_1 and N_2 are arbitrary constants.

From the way in which $S(\tau)$ and $C(\tau)$ were derived, it is readily seen that the coefficients of $\sin(j+1)\tau$ and $\cos(j+1)\tau$ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal.

§ 10. *Construction of the Solution for the Vertical Motion.*

Let

$$\xi = \sum_{j=1}^{\infty} \xi_j \epsilon^j, \tag{77}$$

where the ξ_j are to be periodic with the period 2π in τ.

Since the initial time was chosen for $\epsilon = 0$ so that $\dot{z}(0) = 0$, it may now be chosen so that $\dot{z}(0) = 0$ when $\epsilon \neq 0$. Now $z = \bar{z} + \xi$, and as $\dot{\bar{z}}(0) = 0$, it follows that $\dot{\xi}(0) = 0$. Hence

$$\dot{\xi}_j(0) = 0, \quad j = 1, \cdots \infty. \tag{78}$$

Let (77) be substituted in (67) and let the resulting equation be denoted by (67′). It is an identity in ϵ, and on equating the coefficients of the same powers of ϵ we obtain sets of differential equations which can be integrated and the various constants of integration can be chosen, as we shall show, so as to satisfy the periodicity and initial conditions.

Two types of series occur in this integration and they are similar to $S(\tau)$ and $C(\tau)$ in (71) and (73), respectively. These series will be denoted by

$$S_j(\tau),\ S^{(j)}(\tau),\ \bar{S}_j(\tau),\ \bar{S}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

or

$$C_j(\tau),\ C^{(j)}(\tau),\ \bar{C}_j(\tau),\ \bar{C}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

according as they are similar to $S(\tau)$ or $C(\tau)$, respectively.

Coefficients of ϵ. When the coefficients of ϵ to the first power are equated in (67′) we obtain

$$\ddot{\xi}_1 + \Theta_1 \dot{\xi}_1 + \Theta_2 \xi_1 = C^{(1)}(\tau). \tag{79}$$

The complementary function of this equation is the same as the solution of the equation of variation, viz.,

$$\xi_1 = n_1^{(1)} S(\tau) + n_2^{(1)} [C(\tau) + \tau K S(\tau)], \tag{80}$$

where $n_1^{(1)}$ and $n_2^{(1)}$ are the arbitrary constants. By employing the method of the variation of parameters to find the complete solution, we have

$$\left.\begin{aligned} \dot{n}_1^{(1)} S(\tau) + \dot{n}_2^{(1)} [C(\tau) + \tau K S(\tau)] &= 0, \\ \dot{n}_1^{(1)} \dot{S}(\tau) + \dot{n}_2^{(1)} [\dot{C}(\tau) + K\{\tau \dot{S}(\tau) + S(\tau)\}] &= C^1(\tau). \end{aligned}\right\} \tag{81}$$

The determinant of the coefficients of $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$ in the above equations is the same as (75), and since it is different from zero equations (79) can be solved for $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$. Thus

$$\left.\begin{aligned} \dot{n}_1^{(1)} &= -(1/\Delta)C^{(1)}(\tau)[C(\tau) + \tau K S(\tau)], \\ \dot{n}_2^{(1)} &= \quad (1/\Delta)C^{(1)}(\tau)\, S(\tau). \end{aligned}\right\} \tag{82}$$

Since the coefficients of the same power of μ in $C^{(1)}(\tau)$ and $C(\tau)$ are sums of cosines of the same multiples of τ, the product $C^{(1)}(\tau)C(\tau)$ will yield, in addition to periodic terms, a constant

$$\mu p_1 = \mu[p_1^{(0)} + p_1^{(1)}\mu + \cdots + p_1^{(j)}\mu^j + \cdots\ ],$$

where $p_1^{(j)}$ are real constants. Then the integration of (82) gives

$$\left.\begin{aligned} n_1^{(1)} &= N_1^{(1)} - [\mu\tau p_1 + S^{(1)}(\tau) + \tau K \bar{C}^{(1)}(\tau)], \\ n_2^{(1)} &= N_2^{(1)} + \bar{C}^{(1)}(\tau), \end{aligned}\right\} \tag{83}$$

where $N_1^{(1)}$ and $N_2^{(1)}$ are the constants of integration. Since the coefficients of the sines and cosines of the highest multiples of τ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal, then it follows that $S^{(1)}(\tau)$ and $\bar{C}^{(1)}(\tau)$ have the same property. When (83) is substituted in (80) the complete solution of (79) then becomes

$$\xi_1 = N_1^{(1)}S(\tau) + N_2^{(1)}[C(\tau) + \tau KS(\tau)] - \mu\tau p_1 S(\tau) + (1/\mu)\bar{C}_1(\tau),$$

where $\bar{C}_1(\tau)$ contains no terms independent of μ. In order that ξ_1 shall be periodic $N_2^{(1)}$ must be given the value

$$N_2^{(1)} = \frac{\mu p_1}{K} = \frac{1}{\mu}\,(\text{power series in } \mu), \tag{84}$$

and from the initial conditions (78) it follows that

$$N_1^{(1)} = 0.$$

Hence

$$\xi_1 = (1/\mu)C_1(\tau), \tag{85}$$

where $C_1(\tau)$, like $\bar{C}_1(\tau)$, contains no terms independent of μ.

Coefficients of ϵ^2. Equating the coefficients of ϵ^2 in (67) gives the differential equation

$$\ddot{\xi}_2 + \Theta_1\dot{\xi}_2 + \Theta_2\xi_2 = (1/\mu)C^{(2)}(\tau). \tag{86}$$

Except for the factor $1/\mu$ in the right member, this equation is similar to (79). The general solution is obtained in the same way as at the preceding step and is found to be

$$\xi_2 = N_1^{(2)} S(\tau) + N_2^{(2)} [C(\tau) + \tau K S(\tau)] + \tau p_2 S(\tau) + (1/\mu) \overline{C}^{(2)}(\tau),$$

where $N_1^{(2)}$ and $N_2^{(2)}$ are the constants of integration, and p_2 is a power series in μ. In order to satisfy the periodicity and initial conditions we must put

$$N_2^{(2)} = \frac{p_2}{K} = \frac{1}{\mu^2} \text{ (power series in } \mu), \; N_1^{(2)} = 0.$$

Hence

$$\xi_2 = (1/\mu^2) C_2(\tau),$$

where $C_2(\tau)$, contains no terms independent of μ.

The remaining steps of the integration can be carried on in the same way and by an induction to the general term it can be shown that

$$\xi_n = (1/\mu^n) C_n(\tau),$$

where $C_n(\tau)$ as at the previous steps, contains no terms independent of μ.

On substituting the solutions for the various ξ_j in (77) we obtain

$$\xi = \sum_{j=1}^{\infty} C_j(\tau) (\epsilon/\mu)^j. \tag{87}$$

By applying Macmillan's theorem, quoted in § 3, it is found that the solution (87) converges for $|\epsilon|$ sufficiently small.

When (87) is substituted in (66), the solution for the vertical motion becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau) (\epsilon/\mu)^j. \tag{88}$$

Since ϵ and μ are both arbitrary we may put $\epsilon = \rho\mu$ and therefore (88) becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau) \rho^j. \tag{89}$$

This solution converges for $|\mu|$ and $|\rho|$ sufficiently small.

§ 10. *The Horizontal Motion.* When the first two equations of (3) are transformed by the substitution

$$t - t_0 = \tfrac{1}{2} \sqrt{(k/g)(1+\delta)}\; \tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

already used in (A), and the value of z obtained in (89) is substituted in 2λ, these differential equations become

$$\left.\begin{aligned} \ddot{x} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots] x = 0, \\ \ddot{y} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots] y = 0, \end{aligned}\right\} \tag{90}$$

where each ϕ_j is a power series in μ with sums of cosines of multiples of τ in the coefficients. The function ϕ_0 has the same value as the coefficient of x in (22).

Now let $\rho = \sigma\mu$. Then the coefficients in (90) can be rearranged as power series in μ and the differential equations take the same form as (22). Hence the solutions are

$$x = L_1 e^{i\beta\tau} v_1 + L_2 e^{-i\beta\tau} v_2, \quad y = M_1 e^{i\beta\tau} v_1 + M_2 e^{-i\beta\tau} v_2, \tag{91}$$

where L_1, L_2, M_1, and M_2 are the constants of integration, and β, v_1, and v_2 are similar in form to a, u_1, and u_2, respectively, of (39), (52), or (60) according to the value of $\sqrt{k/p}$.

When the initial values $\dot{x}(0) = y(0) = 0$ are imposed on (91), it follows that

$$[i\beta + \dot{v}_1(0)]\,[L_1 - L_2] = 0, \quad M_1 + M_2 = 0.$$

Since $i\beta + \dot{v}_1(0) \neq 0$ for $|\mu|$ sufficiently small, then

$$L_1 = L_2 = L, \text{ say,}$$

and, from the second equation,

$$M_1 = -M_2 = M, \text{ say.}$$

Hence the solutions (91) become

$$x = L[e^{i\beta\tau} v_1 + e^{-i\beta\tau} v_2], \quad y = M[e^{i\beta\tau} v_1 - e^{-i\beta\tau} v_2].$$

The constants L and M can be determined as in § 7, and it is found that L is real while M is purely imaginary. Suppose $M = iN$. Then

$$x = 2L[\cos\beta\tau\{1 + \sum_{j=1}^{\infty}\sum_{l=0}^{j} \mu^j a_l^{(j)} \cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty}\sum_{l=1}^{j} \mu^j b_l^{(j)} \sin l\tau\}],$$

$$y = 2N[\sin\beta\tau\{1 + \sum_{j=1}^{\infty}\sum_{l=0}^{j} \mu^j a_l^{(j)} \cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty}\sum_{l=1}^{j} \mu^j b_l^{(j)} \sin l\tau\}],$$

where $a_l^{(j)}$ and $b_l^{(j)}$ are real constants. These equations together with equation (89) are the solutions of the equations of motion of the problem under consideration.

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